SIAM REV.
Vol. 39, No. 1, pp. 18–53, March 1997

ON TWO WAYS OF STABILIZING THE HIERARCHICAL BASIS MULTILEVEL METHODS*

PANAYOT S. VASSILEVSKI†

Abstract. A survey of two approaches for stabilizing the hierarchical basis (HB) multilevel preconditioners, both additive and multiplicative, is presented. The first approach is based on the algebraic extension of the two-level methods, exploiting recursive calls to coarser discretization levels. These recursive calls can be viewed as inner iterations (at a given discretization level), exploiting the already defined preconditioner at coarser levels in a polynomially-based inner iteration method. The latter gives rise to hybrid-type multilevel cycles. This is the so-called (hybrid) algebraic multilevel iteration (AMLI) method. The second approach is based on a different direct multilevel splitting of the finite element discretization space. This gives rise to the so-called wavelet multilevel decomposition based on L^2-projections, which in practice must be approximated. Both approaches—the AMLI one and the one based on approximate wavelet decompositions—lead to optimal relative condition numbers of the multilevel preconditioners.

Key words. multilevel methods, hierarchical basis, optimal order preconditioners, algebraic multilevel iteration (AMLI) methods, approximate wavelets

AMS subject classifications. Primary, 65N30; Secondary, 65F10

PII. S0036144595282211

1. Introduction. This paper presents a comprehensive survey of the multilevel methods, i.e., methods that exploit direct decompositions (that is, consisting of nonoverlapping coordinate spaces) of the given finite element discretization space. To be specific, we consider a finite element space $V = V_J$ obtained by successive steps of uniform refinement of an initial coarse triangulation $\mathcal{T}_0$. We denote by $\mathcal{T}_k$ the kth-level triangulation and by V_k the corresponding kth-level discretization space, $k = 0, 1, \dots, J$. We consider here standard conforming piecewise polynomial finite element spaces. This in particular implies that $V_{k-1} \subset V_k$, i.e., that we have a nested sequence of discretization spaces. Finally, we are interested in the following model second-order elliptic bilinear form:

$$A(u, \psi) \equiv \int_\Omega \mathcal{A}\nabla u \cdot \nabla \psi, \quad \text{where } u, \psi \in H_0^1(\Omega). \tag{1}$$

Here Ω is a plane polygon or a three-dimensional (3-d) polytope and $H_0^1(\Omega)$ is the standard Sobolev space of $L^2(\Omega)$-functions vanishing on the boundary of Ω that have all first derivatives also in $L^2(\Omega)$. The given coefficient matrix $\mathcal{A} = (a_{i,j}(x))$, $x \in \Omega$, is symmetric with measurable and bounded entries in Ω, and it is also assumed that $\mathcal{A}$ is positive definite uniformly in Ω.

For the finite element spaces we also assume that V_k admit a Lagrangian (nodal) basis $\{\phi_i^{(k)}\}$, where any index i corresponds to a node x_i which runs over all the degrees of freedom in $\mathcal{N}_k$, the node set at the kth discretization level defined from the triangulation $\mathcal{T}_k$. We denote by h_k the kth discretization level mesh size. We assume that $h_k = 2^{-k}h_0$.

*Received by the editors February 27, 1995; accepted for publication (in revised form) March 28, 1996. This research was partly supported by Bulgarian Ministry of Education, Science and Technology grant MM-415, 1994 and also in part by National Science Foundation grant NSF-INT 95-06184.

http://www.siam.org/journals/sirev/39-1/28221.html

†Center of Informatics and Computing Technology, Bulgarian Academy of Sciences, "Acad. G. Bontchev" Street, Block 25A, 1113 Sofia, Bulgaria (panayot@iscbg.acad.bg).

We are interested in the following variationally-posed boundary value problem.

PROBLEM OF MAIN INTEREST. *Given $f \in L^2(\Omega)$, find $u \in H_0^1(\Omega)$ such that*

$$A(u, \psi) = (f, \psi) \quad \textit{for all } \psi \in H_0^1(\Omega). \tag{2}$$

Here and in what follows we denote by $(.,.)$ and $\|.\|_0$ the standard L^2-inner product and the corresponding L^2-norm, respectively.

The remainder of the paper deals with the following topics:

- classical two-level hierarchical basis (HB) methods; the strengthened Cauchy inequality,
- the HB multilevel methods; additive and multiplicative preconditioning schemes,
- stabilizing the HB method, I: the algebraic multilevel iteration (AMLI) method,
- stabilizing the HB method, II: approximate wavelets.

The main goal of this survey is to present in a compact form how far one could go in developing efficient multilevel preconditioning techniques for solving problem (2) exploiting direct (or, equivalently, hierarchical) decompositions of the finite element discretization space V. Here, $V = V_J$ corresponds to the finest triangulation $\mathcal{T} = \mathcal{T}_J$ which is obtained by $J \geq 1$ successive steps of refinement of the initial (coarse) triangulation $\mathcal{T}_0$.

It is demonstrated in the present paper that using the two approaches described in a number of papers can lead to optimal or practically optimal order methods for both 2-d and 3-d problem domains.

The alternative is to consider decompositions of the fine discretization space V consisting of overlapping components. The latter can lead, for example, to the classical multigrid (MG) methods or to the overlapping Schwarz methods. For an overview of these methods we refer to the book of Bramble [13] and the survey papers of Xu [42], Yserentant [44], Chan and Mathew [15], Dryja, Smith, and Widlund [17], and Griebel and Oswald [19]. For a classical exposition of MG methods we refer to Hackbusch [22].

The presentation in this paper is based on the papers of Bank and Dupont [9], Axelsson and Gustafsson [2], Yserentant [43], Bank, Dupont, and Yserentant [10], Xu [42], Vassilevski [37], [38], Axelsson and Vassilevski [5], [6], [8], and Vassilevski and Wang [41].

2. Classical two-level HB methods; strengthened Cauchy inequality. Here we survey the classical two-level method as proposed by Bank and Dupont [9] (see also Braess [12]) and studied further by Axelsson and Gustafsson [2]. It is a basic step of introducing the multilevel preconditioners.

Consider our bilinear form (1),

$$A(u, v) \equiv \int_\Omega \sum a_{i,j}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \ ,$$

$$u, v \in V \subset H_0^1(\Omega).$$

Given a direct decomposition of the space V,

$$V = V_1 + V_2$$

with coordinate subspaces V_1, V_2. We call this decomposition stable if there exists a constant $\gamma \in [0,1)$ such that

$$A(v_1, v_2) \leq \gamma \, [A(v_1, v_1)]^{\frac{1}{2}} \, [A(v_2, v_2)]^{\frac{1}{2}} \quad \text{for all } v_1 \in V_1, \; v_2 \in V_2. \tag{3}$$

Note that if $\gamma = 0$ the above decomposition is A-orthogonal. In practice we are interested in a constant $\gamma \in [0,1)$ that is independent of the degrees of freedom of V_1 and V_2 (or of their respective mesh parameters h_1 and h_2).

The discretized version of the problem of main interest (2) reads as follows:

For any given right-hand side function $f \in L^2(\Omega)$ find $u_h \in V$ such that

$$A(u_h, \phi) = (f, \phi) \quad \text{for all } \phi \in V.$$

Given also computational bases of $\{\phi_i^{(1)}\}$ of V_1 and $\{\phi_i^{(2)}\}$ of V_2, the above discrete problem takes the following block matrix form:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} \begin{matrix} \} \, V_1 \\ \} \, V_2 \end{matrix} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix} \begin{matrix} \} \, V_1 \\ \} \, V_2 \end{matrix} .$$

Here we seek the solution decomposed as $u_h = u_1 + u_2$, $u_1 \in V_1$, and $u_2 \in V_2$. The respective coefficient vectors of u_1 and u_2 with respect to the given computational bases $\{\phi_i^{(1)}\}$ and $\{\phi_i^{(2)}\}$ are above denoted by $\mathbf{u}_1$ and $\mathbf{u}_2$, respectively. The blocks of the stiffness matrix then read as follows:

$$\begin{aligned} A_{11} &= \{A(\phi_j^{(1)}, \phi_i^{(1)})\}, \\ A_{21} &= \{A(\phi_j^{(1)}, \phi_i^{(2)})\}, \\ A_{12} &= \{A(\phi_j^{(2)}, \phi_i^{(1)})\} \\ &= A_{21}^T, \\ A_{22} &= \{A(\phi_j^{(2)}, \phi_i^{(2)})\}. \end{aligned}$$

The classical two-level preconditioning schemes read as follows:

Given two preconditioners (approximations)

$$M_{11} \text{ to } A_{11}$$

and

$$M_{22} \text{ to } A_{22} \quad (\text{or to } S \equiv A_{22} - A_{21} A_{11}^{-1} A_{12}),$$

one then defines the following.

DEFINITION 1 (multiplicative or block Gauss–Seidel preconditioning scheme).

$$M = \begin{bmatrix} M_{11} & 0 \\ A_{21} & M_{22} \end{bmatrix} \begin{bmatrix} I & M_{11}^{-1} A_{12} \\ 0 & I \end{bmatrix}.$$

It is clear that to implement one inverse action of M one needs two inverse actions of M_{11} and one inverse action of M_{22} in addition to matrix–vector products with the (sparse in practice) matrix blocks A_{21} and A_{12}.

DEFINITION 2 (additive or block Jacobi preconditioning scheme).

$$M_D = \begin{bmatrix} M_{11} & 0 \\ 0 & M_{22} \end{bmatrix}.$$

To implement one action of M_D^{-1} one needs the inverse actions of M_{11} and M_{22}.

There is one more way to define a two-level multiplicative (or product) preconditioning scheme; cf. Bank and Dupont [9].

DEFINITION 3 (block Gauss–Seidel-type preconditioning scheme). *Consider the following splitting:*

$$A_{11} = D_{11} + L_{11} + L_{11}^T$$

with L_{11} a strictly lower triangular part of A_{11} and D_{11} a symmetric positive-definite part of A_{11}. We also assume that D_{11} is a simple matrix, i.e., that it is an easy-to-factor or to-solve-systems-with matrix. For example, D_{11} can be the diagonal of A_{11}. Also, let B_{22} be a preconditioner for A_{22}. Then the two-level block Gauss–Seidel-type preconditioner B is defined as follows:

$$B = \begin{bmatrix} L_{11} + D_{11} & 0 \\ A_{21} & I \end{bmatrix} \begin{bmatrix} D_{11}^{-1} & 0 \\ 0 & B_{22} \end{bmatrix} \begin{bmatrix} L_{11}^T + D_{11} & A_{12} \\ 0 & I \end{bmatrix}.$$

Note that in the case $L_{11} = 0$, i.e., $D_{11} = A_{11}$, B is a special case of the preconditioner defined in Definition 1. It is clear that to implement one inverse action of B one must solve two systems with D_{11} and one system of equations with B_{22} in addition to some eliminations with the (sparse in practice) blocks A_{21}, A_{12}, L_{11}, and L_{11}^T.

We first formulate the following classical result concerning the two-level preconditioners from Definitions 1 and 2.

THEOREM 1 (see Axelsson and Gustafsson [2]). *Assume that*

$$\mathbf{v}_1^T A_{11} \mathbf{v}_1 \le \mathbf{v}_1^T M_{11} \mathbf{v}_1 \le (1+\delta_1) \mathbf{v}_1^T A_{11} \mathbf{v}_1 \text{ for all } \mathbf{v}_1,$$
$$\mathbf{v}_2^T A_{22} \mathbf{v}_2 \le \mathbf{v}_2^T M_{22} \mathbf{v}_2 \le (1+\delta_2) \mathbf{v}_2^T A_{22} \mathbf{v}_2 \text{ for all } \mathbf{v}_2$$

for some nonnegative constants δ_1 and δ_2. Then the following spectral equivalence relations hold:

$$\mathbf{v}^T A \mathbf{v} \le \mathbf{v}^T M \mathbf{v} \le \frac{1}{1-\gamma^2} \left\{ 1 + \frac{1}{2} \left[\delta_1 + \delta_2 + \sqrt{(\delta_1 - \delta_2)^2 + 4\delta_1 \delta_2 \gamma^2} \right] \right\} \mathbf{v}^T A \mathbf{v} \text{ for all } \mathbf{v}.$$

Similarly, for the block diagonal (Jacobi) preconditioner we have

$$(1-\gamma)\Delta_0 \mathbf{v}^T M_D \mathbf{v} \le \mathbf{v}^T A \mathbf{v} \le (1+\gamma) \mathbf{v}^T M_D \mathbf{v} \quad \text{for all } \mathbf{v}.$$

Here,

$$\Delta_0 = \frac{2(1+\gamma)}{1+\delta_2} \left[1 + \Delta + \sqrt{(\Delta - 1)^2 + 4\Delta\gamma^2} \right]^{-1}, \quad \Delta = \frac{1+\delta_1}{1+\delta_2}.$$

Proof. The proof relies on the strengthened Cauchy inequality (3), the spectral equivalence relations between A_{11} and M_{11} and between A_{22} and M_{22}, and on the elementary inequality $2ab \le \xi^{-1} a^2 + \xi b^2$ for an appropriate choice of $\xi > 0$.

For the multiplicative preconditioner M one has

$$\mathbf{v}^T (M - A) \mathbf{v} = \mathbf{v}_1^T (M_{11} - A_{11}) \mathbf{v}_1 + \mathbf{v}_2^T (M_{22} - A_{22}) \mathbf{v}_2 + \mathbf{v}_2^T A_{21} B_{11}^{-1} A_{12} \mathbf{v}_2.$$

This implies the desired left-hand side spectral bound since all terms are nonnegative by assumption. For the upper bound one gets

$$\begin{aligned}
\mathbf{v}^T(M-A)\mathbf{v} &\le \mathbf{v}_1^T(M_{11}-A_{11})\mathbf{v}_1 + \mathbf{v}_2^T(M_{22}-A_{22})\mathbf{v}_2 + \mathbf{v}_2^T A_{21}A_{11}^{-1}A_{12}\mathbf{v}_2 \\
&\le \delta_1 \mathbf{v}_1^T A_{11}\mathbf{v}_1 + \delta_2 \mathbf{v}_2^T A_{22}\mathbf{v}_2 + \gamma^2 \mathbf{v}_2^T A_{22}\mathbf{v}_2 \\
&= \frac{\delta_1}{1-\zeta^{-1}\gamma}(1-\zeta^{-1}\gamma)\mathbf{v}_1^T A_{11}\mathbf{v}_1 + \frac{\delta_2}{1-\zeta\gamma}(1-\zeta\gamma)\mathbf{v}_2^T A_{22}\mathbf{v}_2 \\
&\quad + \frac{\gamma^2}{1-\gamma^2}\mathbf{v}^T A\mathbf{v} \\
&\le \left[\min_{\zeta\in[\gamma,\gamma^{-1}]} \max\left\{\frac{\delta_1}{1-\zeta^{-1}\gamma},\ \frac{\delta_2}{1-\zeta\gamma}\right\} + \frac{\gamma^2}{1-\gamma^2}\right]\mathbf{v}^T A\mathbf{v}.
\end{aligned}$$

Here we have used the inequality (a corollary to the strengthened Cauchy inequality (3))

$$\mathbf{v}^T A\mathbf{v} \ge (1-\gamma\zeta)\mathbf{v}_2^T A_{22}\mathbf{v}_2 + (1-\gamma\zeta^{-1})\mathbf{v}_1^T A_{11}\mathbf{v}_1, \tag{4}$$

valid for any $\zeta \in [\gamma, \gamma^{-1}]$. We also used the same inequality for $\zeta = \gamma$.

Now choosing ζ such that

$$\frac{\delta_1}{1-\zeta^{-1}\gamma} = \frac{\delta_2}{1-\zeta\gamma},$$

i.e., $\zeta = \frac{\delta_2-\delta_1+\sqrt{(\delta_2-\delta_1)^2+4\delta_1\delta_2\gamma^2}}{2\gamma\delta_2}$, one gets

$$\frac{\delta_1}{1-\zeta^{-1}\gamma} = \frac{\delta_1+\delta_2+\sqrt{(\delta_2-\delta_1)^2+4\delta_1\delta_2\gamma^2}}{2(1-\gamma^2)} \le \frac{\delta_1+\delta_2}{1-\gamma^2},$$

which implies the desired estimate.

The additive preconditioner M_D is analyzed in a similar way. We have

$$\begin{aligned}
\mathbf{v}^T A\mathbf{v} &= \mathbf{v}_1^T A_{11}\mathbf{v}_1 + \mathbf{v}_2^T A_{22}\mathbf{v}_2 + 2\mathbf{v}_1^T A_{12}\mathbf{v}_2 \\
&\le (1+\gamma)\left[\mathbf{v}_1^T A_{11}\mathbf{v}_1 + \mathbf{v}_2^T A_{22}\mathbf{v}_2\right] \\
&\le (1+\gamma)\left[\mathbf{v}_1^T M_{11}\mathbf{v}_1 + \mathbf{v}_2^T M_{22}\mathbf{v}_2\right] \\
&= (1+\gamma)\mathbf{v}^T M_D\mathbf{v}.
\end{aligned}$$

For the estimate from below, one has

$$\begin{aligned}
\mathbf{v}^T A\mathbf{v} &\ge (1-\zeta\gamma)\mathbf{v}_1^T A_{11}\mathbf{v}_1 + (1-\zeta^{-1}\gamma)\mathbf{v}_2^T A_{22}\mathbf{v}_2 \\
&\ge \frac{1-\zeta^{-1}\gamma}{1+\delta_2}\mathbf{v}_2^T M_{22}\mathbf{v}_2 + \frac{1-\zeta\gamma}{1+\delta_1}\mathbf{v}_1^T M_{11}\mathbf{v}_1 \\
&\ge \max_{\zeta\in[\gamma,\gamma^{-1}]} \min\left\{\frac{1-\zeta^{-1}\gamma}{1+\delta_2},\ \frac{1-\zeta\gamma}{1+\delta_1}\right\}\mathbf{v}^T M_D\mathbf{v}.
\end{aligned}$$

The parameter $\zeta \in [\gamma, \gamma^{-1}]$ is chosen such that

$$\frac{1-\zeta^{-1}\gamma}{1+\delta_2} = \frac{1-\zeta\gamma}{1+\delta_1},$$

or letting $\Delta = \frac{1+\delta_1}{1+\delta_2}$, we have the quadratic equation $\gamma\zeta^2 - (1-\Delta)\zeta - \Delta\gamma = 0$ for ζ. This gives

$$\zeta = \frac{1-\Delta+\sqrt{(\Delta-1)^2+4\gamma^2\Delta}}{2\gamma}.$$

Thus the desired left-hand side estimate becomes $\mathbf{v}^T A\mathbf{v} \geq \Delta_0(1-\gamma)\mathbf{v}^T M_D \mathbf{v}$ with

$$\begin{aligned}\Delta_0 &= \frac{1-\gamma\zeta}{(1+\delta_1)(1-\gamma)} = \frac{2\Delta(1+\gamma)}{1+\delta_1}\left\{1+\Delta+\sqrt{(\Delta-1)^2+4\Delta\gamma^2}\right\}^{-1} \\ &\geq \frac{2\Delta}{1+\delta_1}\{1+\Delta+|\Delta-1|\}^{-1} \\ &= \frac{1}{1+\max\{\delta_1,\delta_2\}}. \qquad \square\end{aligned}$$

For the two-level preconditioner B from Definition 3 the following well-known result holds; compare, e.g., Bank and Dupont [9] (see also Bank, Dupont, and Yserentant [10]).

THEOREM 2. *Assume that*

$$\mathbf{v}_2^T A_{22}\mathbf{v}_2 \leq \mathbf{v}_2^T B_{22}\mathbf{v}_2 \leq (1+b_2)\mathbf{v}_2^T A_{22}\mathbf{v}_2 \quad \textit{for all } \mathbf{v}_2$$

for some constant $b_2 \geq 0$. Then the following spectral equivalence relations hold:

$$\mathbf{v}^T A\mathbf{v} \leq \mathbf{v}^T B\mathbf{v} \leq \kappa_{TL}\mathbf{v}^T A\mathbf{v} \quad \textit{for all } \mathbf{v},$$

where the constant κ_{TL} depends only on γ and b_2, on the spectral condition number of $D_{11}^{-1}A_{11}$, and on the (standard spectral) norm of $D_{11}^{-1/2}L_{11}D_{11}^{-1/2}$ (the same as of $D_{11}^{-1/2}L_{11}^T D_{11}^{-1/2}$), which is defined for any matrix G by $\|G\|^2 = \sup_{\mathbf{w}} \frac{\mathbf{w}^T G^T G\mathbf{w}}{\mathbf{w}^T\mathbf{w}}$.

Proof. Let $\lambda\left[D_{11}^{-1}A_{11}\right] \in [\sigma_1^{-1}, \sigma_2]$ and denote $\ell = \|D_{11}^{-1/2}L_{11}^T D_{11}^{-1/2}\|$. The left-hand side of the desired inequality is seen from the identity

(5)
$$\begin{aligned} B - A &= \begin{bmatrix} (L_{11}+D_{11})D_{11}^{-1}(L_{11}^T+D_{11}) - A_{11} & (L_{11}+D_{11})D_{11}^{-1}A_{12} - A_{12} \\ A_{21}D_{11}^{-1}(L_{11}^T+D_{11}) - A_{21} & B_{22} - A_{22} + A_{21}D_{11}^{-1}A_{12} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & B_{22}-A_{22} \end{bmatrix} + \begin{bmatrix} L_{11}D_{11}^{-1} & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} D_{11} & A_{12} \\ A_{21} & A_{21}D_{11}^{-1}A_{12} \end{bmatrix}\begin{bmatrix} D_{11}^{-1}L_{11}^T & 0 \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & B_{22}-A_{22} \end{bmatrix} + \begin{bmatrix} L_{11} & 0 \\ A_{21} & I \end{bmatrix}\begin{bmatrix} D_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} L_{11}^T & A_{12} \\ 0 & I \end{bmatrix}, \end{aligned}$$

noting that both last terms are positive semidefinite.

The right-hand side inequality is seen again from the last identity (5) and the following corollaries from the strengthened Cauchy inequality (letting $\zeta = \gamma^{-1}$ and $\zeta = \gamma$, respectively, in (4))

$$\mathbf{v}_1^T A_{11}\mathbf{v}_1 \leq \frac{1}{1-\gamma^2}\mathbf{v}^T A\mathbf{v} \quad \text{for all } \mathbf{v} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix}$$

and

$$\mathbf{v}_2^T A_{22} \mathbf{v}_2 \le \frac{1}{1-\gamma^2} \mathbf{v}^T A \mathbf{v} \quad \text{for all } \mathbf{v} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix}.$$

The spectral equivalence relations between B_{22} and A_{22} and A_{11} and D_{11} and the norm estimate of $D_{11}^{-1/2} L_{11} D_{11}^{-1/2}$ are also used.

Following the classical result for the convergence factor of the symmetric block Gauss–Seidel preconditioner $B_{11} \equiv (D_{11} + L_{11}) D_{11}^{-1} (D_{11} + L_{11}^T)$ one has

$$\mathbf{v}_1^T A_{11} \mathbf{v}_1 \le \mathbf{v}_1^T B_{11} \mathbf{v}_1 \le (1 + b_1) \mathbf{v}_1^T A_{11} \mathbf{v}_1 \quad \text{for all } \mathbf{v}_1,$$

where $b_1 \le \ell^2 \sigma_1$.

Identity (5) implies for any $\mathbf{v} = [\begin{smallmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{smallmatrix}]$ (letting $\mathbf{w}_1 = D_{11}^{-1} L_{11}^T \mathbf{v}_1$) that

$$\begin{aligned}
\mathbf{v}^T(B-A)\mathbf{v} &= \mathbf{v}_2^T (B_{22} - A_{22}) \mathbf{v}_2 + \mathbf{w}_1^T D_{11} \mathbf{w}_1 + 2\mathbf{w}_1^T A_{12} \mathbf{v}_2 + \mathbf{v}_2^T A_{21} B_{11}^{-1} A_{12} \mathbf{v}_2 \\
&\le b_2 \mathbf{v}_2^T A_{22} \mathbf{v}_2 + \gamma^2 \mathbf{v}_2^T A_{22} \mathbf{v}_2 + \mathbf{w}_1^T D_{11} \mathbf{w}_1 + \gamma\zeta \mathbf{w}_1^T A_{11} \mathbf{w}_1 \\
&\quad + \gamma\zeta^{-1} \mathbf{v}_2^T A_{22} \mathbf{v}_2 \\
&\le (b_2 + \gamma\zeta^{-1}) \mathbf{v}_2^T A_{22} \mathbf{v}_2 + (\sigma_2\gamma\zeta + 1) b_1 \mathbf{v}_1^T A_{11} \mathbf{v}_1 + \gamma^2 \mathbf{v}_2^T A_{22} \mathbf{v}_2 \qquad (6) \\
&\le \frac{1}{1-\gamma} \min_{\zeta \in (0,\infty)} \max \left\{ b_2 + \gamma\zeta^{-1},\ (\sigma_2\gamma\zeta + 1) b_1 \right\} \mathbf{v}^T A \mathbf{v} \\
&\quad + \left[\frac{1}{1-\gamma^2} - 1 \right] \mathbf{v}^T A \mathbf{v}.
\end{aligned}$$

Now choose $\zeta > 0$ such that $b_2\zeta + \gamma = b_1(\sigma_2\gamma\zeta + 1)\zeta$; i.e.,

$$\zeta = \frac{b_2 - b_1 + \sqrt{(b_1 - b_2)^2 + 4\sigma_2\gamma^2 b_1}}{2 b_1 \sigma_2 \gamma}.$$

Substituting this value of ζ in (6), the following upper bound for κ_{TL} is obtained:

$$\kappa_{TL} \le \frac{1}{1-\gamma^2} + \frac{1}{2(1-\gamma)} \left(b_2 + b_1 + \sqrt{(b_1 - b_2)^2 + 4\sigma_2\gamma^2 b_1} \right). \qquad \square$$

One typical classical example of the two-level preconditioning scheme is based on the two-level hierarchical basis. Consider a finite element space $V = V_h$ that corresponds to a quasi-uniform triangulation $\mathcal{T} = \mathcal{T}_h$ obtained by a fixed number of successive steps of uniform refinement of an initial (coarse) quasi-uniform triangulation $\tilde{\mathcal{T}} = \mathcal{T}_H$ and let $\tilde{V} = V_H$ $(= V_2)$ be the corresponding coarse finite element space. Note that $\tilde{V} \subset V$. Now introduce the nodal interpolation operator $\Pi = \Pi_H$ defined for any continuous function v as follows: $(\Pi v)(x) = v(x)$, where x runs over all nodal degrees of freedom of the coarse triangulation $\tilde{\mathcal{T}} = \mathcal{T}_H$ $(= \mathcal{T}_2)$. Then the following stable and direct decomposition of V is of interest:

$$V = \tilde{V} + (I - \Pi)V.$$

We let $V_1 \equiv (I - \Pi)V$ and $V_2 = \tilde{V}$. It is well known that the following strengthened Cauchy inequality holds (compare, e.g., Bank and Dupont [9], Maitre and Musy [24], or Axelsson and Gustafsson [2]):

$$A(v_1, \tilde{v}) \le \gamma \left[A(v_1, v_1) \right]^{\frac{1}{2}} \left[A(\tilde{v}, \tilde{v}) \right]^{\frac{1}{2}} \quad \text{for all } v_1 \in V_1 = (I - \Pi)V \quad \text{and all } \tilde{v} \in \tilde{V}.$$

The constant $\gamma = \max_{T \in \tilde{\mathcal{T}}} \gamma_T$, where $\gamma_T = \sup_{v_1 \in V_1,\, v_2 \in V_2} \frac{A_T(v_1,v_2)}{\sqrt{A_T(v_1,v_1)}\sqrt{A_T(v_2,v_2)}}$ and $A_T(.,.)$ is the restriction of A to the elements $T \in \mathcal{T}_H$. This means that $\gamma \in [0,1)$ can be estimated locally. Explicit expressions and/or numerical estimates of γ_T are derived in Maitre and Musy [24], Achchab and Maitre [1], Axelsson and Gustafsson [2], Vassilevski and Etova [39], Margenov [25], Margenov, Xanthis, and Zikatanov [27], Eijkhout and Vassilevski [18], and others for various finite element spaces and bilinear forms A.

There is an equivalent form of the strengthened Cauchy inequality; namely, consider the norm estimate of the local projection operator Π,

$$A(\Pi v, \Pi v) \leq \eta A(v,v) \quad \text{for all } v \in V.$$

Then $\gamma = \sqrt{1 - \frac{1}{\eta}}$. This is seen from the following inequality:

$$A(\Pi v, \Pi v) \leq \eta A(v_t, v_t),$$

where $v_t = \Pi v + t(I - \Pi)w$ for any real number t and arbitrary v and w, since $\Pi v_t = \Pi^2 v + t\Pi(I-\Pi)w = \Pi v$. The latter is true since $\Pi^2 = \Pi$. This implies the positive semidefiniteness of the quadratic form $t^2 A((I-\Pi)w,(I-\Pi)w) + 2tA(\Pi v, (I-\Pi)w) + (1-\eta^{-1})A(\Pi v, \Pi v)$, which implies that its discriminant is nonnegative, and this is precisely the strengthened Cauchy inequality

$$(A(v_1,v_2))^2 \leq \left(1 - \frac{1}{\eta}\right) A(v_1,v_1)A(v_2,v_2) \text{ for } v_1 = (I-\Pi)w \in V_1 \text{ and } v_2 = \Pi v \in V_2.$$

The above equivalence was established in Vassilevski [38]. It is well known that for the nodal interpolation operator Π the above norm bound η depends on $\frac{H}{h}$; i.e., $\eta = \eta\left(\frac{H}{h}\right)$ (see (10) below). Hence if $\frac{H}{h} \leq C$ the constant γ will remain bounded away from unity uniformly with respect to $h \to 0$.

There is another important feature of the two-level block form of the resulting stiffness matrix A computed from the two-level HB of V; namely, using the nodal basis of the coarse space $\tilde{V} = V_H$ and the nodal basis of V_1 (the hierarchical complement of $\tilde{V}$ in V), the first block A_{11} of the stiffness block matrix is well conditioned (note that we have assumed that $\frac{H}{h} \leq C$). Hence A_{11} allows for good approximations. A computationally feasible approximation is a properly scaled (also done element by element with respect to the elements of $\mathcal{T}_H$) diagonal part of A_{11}. This in particular shows that D_{11} (the scalar diagonal part of A_{11}) is spectrally equivalent to A_{11} and the corresponding spectral equivalence constants can be estimated locally. Similarly, the spectral norm of $D_{11}^{-1/2} L_{11} D_{11}^{-1/2}$ (for L_{11} see Definition 3) can also be estimated locally. This norm takes part in the estimates in Theorem 2. In some cases, e.g., when bisection refinement is used (cf. Mitchell [29] and also Maubach [28] including 3-d elements), A_{11} itself is diagonal and hence no further approximation of A_{11} is needed.

For the case of rough coefficients (discontinuous or in the presence of anisotropy) one must take special care of how to approximate A_{11}. Some possibilities are found in Margenov and Vassilevski [26]; see also Margenov, Xanthis, and Zikatanov [27]. We next note that the second block A_{22} is the stiffness matrix $\tilde{A} \equiv A_H$ computed from the coarse space V_H. It can be approximated by any available preconditioner for the coarse grid problem. One possibility is also to nest successively the same two-level procedure and thus to end up with a multilevel HB preconditioning scheme. Another possibility is just to use a more classical (block) ILU method (if the coarse mesh is not too fine).

3. The HB multilevel method; additive and multiplicative preconditioning schemes. The straightforward extension of the two-level HB method by successively nesting the two-level scheme does not lead to optimal order methods. For 2-d problems, as proposed in Yserentant [43] and Bank, Dupont, and Yserentant [10], this gives satisfactory nearly optimal preconditioning methods. For 3-d problems this is not as attractive; see, e.g., Ong [32].

To define the multilevel HB preconditioning methods one first defines the nodal interpolation operators Π_k defined for any continuous function v as follows: $(\Pi_k v)(x) = v(x)$, where x runs over all nodal degrees of freedom in the kth-level triangulation $\mathcal{T}_k \equiv \mathcal{T}_{h_k}$, $h_k = \frac{1}{2}h_{k-1}$, and $h_0 = H$ is the mesh size of the initial (coarse) triangulation. The elements of $\mathcal{T}_k$ are obtained by uniformly refining each element of $\mathcal{T}_{k-1}$ into four congruent ones (in two dimensions).

To analyze the multilevel methods under discussion it is more convenient to use the HB of V which is defined by induction as follows. Assume that the HB of V_{k-1} has been defined. Then the HB of V_k is defined on the basis of the direct decomposition of $V_k = V_{k-1} + (I - \Pi_{k-1})V_k$ by keeping the HB of V_{k-1} and adding to it the nodal basis functions of V_k that correspond to the two-level hierarchical complement $V_k^{(1)} \equiv (I - \Pi_{k-1})V_k$ of V_{k-1} in V_k.

At discretization level k the HB stiffness matrix $A^{(k)}$ computed from $A(.,.)$ and the HB of V_k admits the following two-level block form:

$$A^{(k)} = \begin{array}{cl} \left[\begin{array}{cc} A_{11}^{(k)} & A_{12}^{(k)} \\ A_{21}^{(k)} & A^{(k-1)} \end{array}\right] & \begin{array}{l} \} \; V_k^{(1)} \\ \} \; V_{k-1} \end{array} \end{array} .$$

Assume now that we have some given symmetric and positive-definite approximations $B_{11}^{(k)}$ to the first blocks $A_{11}^{(k)}$ on the diagonal of $A^{(k)}$. Let the following spectral equivalence relations hold:

$$\mathbf{v}_1^T A_{11}^{(k)} \mathbf{v}_1 \le \mathbf{v}_1^T B_{11}^{(k)} \mathbf{v}_1 \le (1 + b_1) \mathbf{v}_1^T A_{11}^{(k)} \mathbf{v}_1 \quad \text{for all } \mathbf{v}_1 \text{ such that } v_1 \in V_k^{(1)}.$$

Here, b_1 is a nonnegative constant. For any function $g \in V_k^{(1)}$ we will denote by $\mathbf{g}_1$ its nodal basis coefficient vector. For any $\tilde{v} \in V_{k-1}$ we will denote by $\tilde{\mathbf{v}}$ its $(k-1)$th-level HB coefficient vector, i.e., using the HB of V_{k-1}.

We can now define the following two multilevel HB preconditioning schemes.

Definition 4 (multiplicative or block Gauss–Seidel HB preconditioning scheme (Vassilevski [37])). *Define $M^{(0)} = A^{(0)}$. For $k \ge 1$ assume that $M^{(k-1)}$, the HB preconditioner for $A^{(k-1)}$, has been defined. Then*

$$M^{(k)} = \left[\begin{array}{cc} B_{11}^{(k)} & 0 \\ A_{21}^{(k)} & M^{(k-1)} \end{array}\right] \left[\begin{array}{cc} I & {B_{11}^{(k)}}^{-1} A_{12}^{(k)} \\ 0 & I \end{array}\right] \begin{array}{l} \} \;\; V_k^{(1)} \\ \} \;\; V_{k-1} \end{array} .$$

Definition 5 (block diagonal or block Jacobi HB preconditioner (Yserentant [43])).

$$M_D^{(k)} = \left[\begin{array}{ccccc} B_{11}^{(k)} & & & & 0 \\ & B_{11}^{(k-1)} & & & \\ & & \ddots & & \\ & & & B_{11}^{(1)} & \\ 0 & & & & A^{(0)} \end{array}\right] \begin{array}{l} \} \;\; V_k^{(1)} \\ \} \;\; V_{k-1}^{(1)} \\ \vdots \\ \} \;\; V_1^{(1)} \\ \} \;\; V_0 \end{array} .$$

DEFINITION 6 (HBMG preconditioner of Bank, Dupont, and Yserentant [10] or a multiplicative or block Gauss–Seidel-type HB preconditioner). *Assume that* $A_{11}^{(k)}$ *is split as*

$$A_{11}^{(k)} = D_{11}^{(k)} + L_{11}^{(k)} + L_{11}^{(k)^T},$$

where $L_{11}^{(k)}$ *is a strictly lower triangular part of* $A_{11}^{(k)}$ *and* $D_{11}^{(k)}$ *is a simple part* $A_{11}^{(k)}$*. That is, we assume that* $D_{11}^{(k)}$ *is an easy-to-factor or to-solve-systems matrix (e.g., the scalar diagonal part of* $A_{11}^{(k)}$*). It is also assumed that* $D_{11}^{(k)}$ *is symmetric and positive definite.*

Define $B^{(0)} = A^{(0)}$*. For* $k \geq 1$ *assume that* $B^{(k-1)}$*, the HBMG preconditioner for* $A^{(k-1)}$*, has been defined. Then*

$$B^{(k)} = \begin{bmatrix} L_{11}^{(k)} + D_{11}^{(k)} & 0 \\ A_{21}^{(k)} & I \end{bmatrix} \begin{bmatrix} D_{11}^{(k)^{-1}} & 0 \\ 0 & B^{(k-1)} \end{bmatrix} \begin{bmatrix} L_{11}^{(k)^T} + D_{11}^{(k)} & A_{12}^{(k)} \\ 0 & I \end{bmatrix} \begin{array}{l} \} \quad V_k^{(1)} \\ \} \quad V_{k-1} \end{array}.$$

The following results hold for 2-d polygonal domains Ω (see Yserentant [43] for the additive preconditioner, Definition 5 and Vassilevski [37] for the multiplicative one from Definition 4).

THEOREM 3.

$$\mathbf{v}^T A^{(k)} \mathbf{v} \leq \mathbf{v}^T M^{(k)} \mathbf{v} \leq (1 + Ck^2) \mathbf{v}^T A^{(k)} \mathbf{v} \quad \textit{for all } \mathbf{v} \textit{ such that } v \in V_k.$$

Similarly,

$$C_1 \mathbf{v}^T A^{(k)} \mathbf{v} \leq \mathbf{v}^T M_D^{(k)} \mathbf{v} \leq C_2 (1 + k^2) \mathbf{v}^T A^{(k)} \mathbf{v} \quad \textit{for all } \mathbf{v} \textit{ such that } v \in V_k.$$

The constants C, C_1*, and* C_2 *are mesh independent (or level independent). Also, these constants are independent of possible large jumps in the coefficients of the bilinear form* $A(.,.)$ *if they occur only across the edges of the elements for the coarsest triangulation* $\mathcal{T}_0$*.*

Proof. The proof of the spectral bounds for the multiplicative preconditioner $M^{(k)}$ is based on the following identity. Given $\mathbf{v} = \mathbf{v}^{(k)}$, the HB coefficient vector of any given function $v \in V_k$, starting with $s = k$ down to 1, one successively defines $\mathbf{v}_1^{(s)}$ as the sth-level nodal coefficient vector of $(\Pi_s - \Pi_{s-1})v \in V_s^{(1)}$ and $\mathbf{v}^{(s-1)} = \mathbf{v}_2^{(s)}$ as the $(s-1)$th-level HB coefficient vector of $\Pi_{s-1} v$. Then the main identity reads as

$$\begin{aligned} \mathbf{v}^T (M^{(k)} - A^{(k)}) \mathbf{v} = \mathbf{v}_1^{(k)^T} (B_{11}^{(k)} - A_{11}^{(k)}) \mathbf{v}_1^{(k)} + \mathbf{v}_2^{(k)^T} (M^{(k-1)} - A^{(k-1)}) \mathbf{v}_2^{(k)} \\ + \mathbf{v}_2^{(k)^T} A_{21}^{(k)} B_{11}^{(k)^{-1}} A_{12}^{(k)} \mathbf{v}_2^{(k)}. \end{aligned} \tag{7}$$

This immediately implies the left-hand side of the required spectral bound since all terms are nonnegative (for the term $\mathbf{v}_2^{(k)^T} (M^{(k-1)} - A^{(k-1)}) \mathbf{v}_2^{(k)}$ this follows by induction recalling that $M^{(0)} = A^{(0)}$).

For the upper bound, using the above identity (7) recursively, one gets

$$\mathbf{v}^T (M^{(k)} - A^{(k)}) \mathbf{v} = \sum_{s=1}^{k} \mathbf{v}_1^{(s)^T} (B_{11}^{(s)} - A_{11}^{(s)}) \mathbf{v}_1^{(s)} + \sum_{s=0}^{k-1} \mathbf{v}^{(s)^T} A_{21}^{(s+1)} B_{11}^{(s+1)^{-1}} A_{12}^{(s+1)} \mathbf{v}^{(s)}.$$

This identity implies the inequalities

$$
\begin{aligned}
\mathbf{v}^T(M^{(k)} - A^{(k)})\mathbf{v} &\le b_1 \sum_{s=1}^{k} \mathbf{v}_1^{(s)^T} A_{11}^{(s)} \mathbf{v}_1^{(s)} + \sum_{s=0}^{k-1} \mathbf{v}^{(s)^T} A_{21}^{(s+1)} A_{11}^{(s+1)^{-1}} A_{12}^{(s+1)} \mathbf{v}^{(s)} \\
&\le b_1 \sum_{s=1}^{k} \mathbf{v}_1^{(s)^T} A_{11}^{(s)} \mathbf{v}_1^{(s)} + \sum_{s=0}^{k-1} \mathbf{v}^{(s)^T} A^{(s)} \mathbf{v}^{(s)}.
\end{aligned} \tag{8}
$$

Here we have used the inequality

$$\mathbf{v}^{(s)^T} A_{21}^{(s+1)} A_{11}^{(s+1)^{-1}} A_{12}^{(s+1)} \mathbf{v}^{(s)} \le \mathbf{v}^{(s)^T} A^{(s)} \mathbf{v}^{(s)} \quad \text{for all } \mathbf{v}^{(s)},$$

which follows from the positive definiteness of the Schur complement $S^{(s+1)}$ of the symmetric positive-definite matrix $A^{(s+1)}$, where $S^{(s+1)} \equiv A^{(s)} - A_{21}^{(s+1)} A_{11}^{(s+1)^{-1}} A_{12}^{(s+1)}$.

To complete the proof we then use the estimates (see (4) with $\zeta = \gamma^{-1}$ and $A = A^{(s)}$)

$$
\begin{aligned}
\mathbf{v}_1^{(s)^T} A_{11}^{(s)} \mathbf{v}_1^{(s)} &\le \frac{1}{1-\gamma^2} \mathbf{v}^{(s)^T} A^{(s)} \mathbf{v}^{(s)} \\
&= \frac{1}{1-\gamma^2} A(\Pi_s v, \Pi_s v) \\
&\le \frac{1}{1-\gamma^2} \eta\left(\frac{h_s}{h_k}\right) A(v,v).
\end{aligned} \tag{9}
$$

The function η represents the energy norm of the nodal interpolation operator Π_s; i.e., for any integers $0 \le s \le k \le J$ there holds

$$A(\Pi_s v, \Pi_s v) \le \eta\left(\frac{h_s}{h_k}\right) A(v,v) \quad \text{for all } v \in V_k.$$

It is well known that η has the following behavior (compare, e.g., Yserentant [43], Ong [32], and Vassilevski [38]) for some mesh-independent constant C:

$$\eta(t) = \begin{cases} 1 + C\log t, & \Omega \quad \text{a 2-d polygon,} \\ 1 + C(t-1), & \Omega \quad \text{a 3-d polytope.} \end{cases} \tag{10}$$

The constant C can be estimated locally with respect to the elements from the initial coarse triangulation $\mathcal{T}_0$ and hence is independent with respect to possible jumps of the entries of coefficient matrix $\mathcal{A}$ as long as this only occurs across edges (faces) of the elements of $\mathcal{T}_0$. In the present case $d = 2$; hence $\eta(\frac{h_s}{h_k}) = C(k-s)+1$. Summing up the last inequalities, (9) leads to the required upper spectral bound. Namely, from (8), (9), and $\eta(\frac{h_s}{h_k}) = C(k-s)+1$ one gets

$$
\begin{aligned}
\mathbf{v}^T M^{(k)} \mathbf{v} &\le b_1 \sum_{s=1}^{k} \mathbf{v}_1^{(s)^T} A_{11}^{(s)} \mathbf{v}_1^{(s)} + \sum_{s=0}^{k} \mathbf{v}^{(s)^T} A^{(s)} \mathbf{v}^{(s)} \\
&\le (1 + Ck)\mathbf{v}^T A^{(k)} \mathbf{v} + \left(\frac{b_1}{1-\gamma^2} + 1\right) \sum_{s=1}^{k} \left[1 + C(k-s)\right] \mathbf{v}^T A^{(k)} \mathbf{v} \\
&\le \left\{1 + \left[C + 1 + \frac{b_1}{1-\gamma^2}\right] k + C\left(1 + \frac{b_1}{1-\gamma^2}\right) \frac{k(k-1)}{2}\right\} \mathbf{v}^T A^{(k)} \mathbf{v} \\
&\le (1 + O(k^2))\mathbf{v}^T A^{(k)} \mathbf{v}.
\end{aligned}
$$

To prove the bounds in the estimates of the eigenvalues ${M_D^{(k)}}^{-1} A^{(k)}$ one proceeds as follows. Given $v \in V_k$ with a kth-level HB coefficient vector $\mathbf{v}$, let $v_s^1 = (\Pi_s - \Pi_{s-1})v$ and $v_s = \Pi_s v$ and denote by $\mathbf{v}_1^{(s)}$ the coefficient vector of v_s^1 and by $\mathbf{v}^{(s)}$ the sth-level HB coefficient vector of v_s. Then

$$\mathbf{v}^T A^{(k)} \mathbf{v} = A(v,v) = A\left(v_0 + \sum_{s=1}^{k} v_s^1, v_0 + \sum_{r=1}^{k} v_r^1\right) \leq 2A(v_0, v_0) + 2\sum_{s,r=1}^{k} A(v_s^1, v_r^1).$$

We now use the following strengthened Cauchy inequality (cf. Yserentant [43]):

$$A(v_s^1, v_r^1) \leq C\delta^{|r-s|}(A(v_s^1, v_s^1))^{\frac{1}{2}}(A(v_r^1, v_r^1))^{\frac{1}{2}}, \tag{11}$$

which holds for a constant $\delta \in (0,1)$ ($\delta = \frac{1}{\sqrt{2}}$ for uniform refinement with $h_s = \frac{1}{2}h_{s-1}$). This immediately shows the estimate

$$\sum_{r,s=1}^{k} A(v_r^1, v_s^1) \leq C\,\frac{1+\delta}{1-\delta}\sum_{s=1}^{k} A(v_s^1, v_s^1).$$

Therefore, one obtains

$$\begin{aligned}
\mathbf{v}^T A^{(k)} \mathbf{v} = A(v,v) &\leq 2C\frac{1+\delta}{1-\delta}\sum_{s=1}^{k} A(v_s^1, v_s^1) + 2A(v_0, v_0) \\
&= 2C\frac{1+\delta}{1-\delta}\sum_{s=1}^{k} {\mathbf{v}_1^{(s)}}^T A_{11}^{(s)} \mathbf{v}_1^{(s)} + 2\mathbf{v}^{(0)} A^{(0)} \mathbf{v}^{(0)} \\
&\leq 2\max\left\{1, C\,\frac{1+\delta}{1-\delta}\right\} \mathbf{v}^T M_D^{(k)} \mathbf{v}.
\end{aligned}$$

We also have

$$\begin{aligned}
\mathbf{v}^T M_D^{(k)} \mathbf{v} &= {\mathbf{v}^{(0)}}^T A^{(0)} \mathbf{v}^{(0)} + \sum_{s=1}^{k} {\mathbf{v}_1^{(s)}}^T B_{11}^{(s)} \mathbf{v}_1^{(s)} \\
&\leq (1+b_1)\sum_{s=1}^{k} {\mathbf{v}_1^{(s)}}^T A_{11}^{(s)} \mathbf{v}_1^{(s)} + {\mathbf{v}^{(0)}}^T A^{(0)} \mathbf{v}^{(0)} \\
&\leq \frac{1+b_1}{1-\gamma^2}\,\sum_{s=1}^{k} {\mathbf{v}^{(s)}}^T A^{(s)} \mathbf{v}^{(s)} + {\mathbf{v}^{(0)}}^T A^{(0)} \mathbf{v}^{(0)} \\
&\leq \frac{1+b_1}{1-\gamma^2}\,\sum_{s=1}^{k} [1 + C(k-s)]\mathbf{v}^T A^{(k)} \mathbf{v} + (1+Ck)\mathbf{v}^T A^{(k)} \mathbf{v} \\
&\leq \left[1 + \left(C + \frac{1+b_1}{1-\gamma^2}\right)k + C\,\frac{1+b_1}{1-\gamma^2}\,\frac{k(k-1)}{2}\right]\mathbf{v}^T A^{(k)} \mathbf{v} \\
&\leq (1+Ck^2)\mathbf{v}^T A^{(k)} \mathbf{v}.
\end{aligned}$$

Note that the latter sum is estimated in the same way as in the case of the multiplicative preconditioner $M^{(k)}$. This completes the proof of the theorem. ∎

The hierarchical basis multigrid (HBMG) preconditioner from Definition 6 of Bank, Dupont, and Yserentant [10] can be analyzed similarly as in Theorem 3. It has the same nearly optimal properties (for planar polygonal domains) as the other two preconditioners from Definition 4 and Definition 5. More specifically, we have Theorem 4.

THEOREM 4. *Consider the HBMG preconditioner $B^{(k)}$ from Definition* 6. *Then the following spectral equivalence relations hold:*

$$\mathbf{v}^T A^{(k)} \mathbf{v} \le \mathbf{v}^T B^{(k)} \mathbf{v} \le (1 + Ck^2) \mathbf{v}^T A^{(k)} \mathbf{v} \quad \textit{for all } \mathbf{v}.$$

The constant $C > 0$ is mesh independent as well as independent of possible jumps in the coefficients of $\mathcal{A}$ as long as these only occur across edges of elements from the initial (coarse) triangulation $\mathcal{T}_0$.

Proof. Use the identity which is derived similarly as (5),

(12)

$$B^{(k)} - A^{(k)} = \begin{bmatrix} 0 & 0 \\ 0 & B^{(k-1)} - A^{(k-1)} \end{bmatrix} + \begin{bmatrix} L_{11}^{(k)} & 0 \\ A_{21}^{(k)} & I \end{bmatrix} \begin{bmatrix} D_{11}^{(k)^{-1}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} L_{11}^{(k)^T} & A_{12}^{(k)} \\ 0 & I \end{bmatrix}.$$

This first shows (by induction since $B^{(0)} = A^{(0)}$) that $B^{(k)} - A^{(k)}$ is positive semidefinite since all terms above are positive semidefinite.

The upper bound of the spectrum of $A^{(k)^{-1}} B^{(k)}$ is obtained based on the above identity (12) being used recursively (the notation is the same as in the proof of Theorem 3); i.e., denoting $B_{11}^{(k)} = (L_{11}^{(k)} + D_{11}^{(k)}) D_{11}^{(k)^{-1}} (D_{11}^{(k)} + L_{11}^{(k)^T})$, one gets

$$\begin{aligned}
\mathbf{v}^T (B^{(k)} - A^{(k)}) \mathbf{v} &\le \mathbf{v}_2^{(k)^T} (B^{(k-1)} - A^{(k-1)}) \mathbf{v}_2^{(k)} + \mathbf{v}_1^{(k)^T} L_{11}^{(k)} D_{11}^{(k)^{-1}} L_{11}^{(k)^T} \mathbf{v}_1^{(k)} \\
&\quad + 2 \mathbf{v}_1^{(k)^T} L_{11}^{(k)} D_{11}^{(k)^{-1}} A_{12}^{(k)} \mathbf{v}_2^{(k)} + \mathbf{v}_2^{(k)^T} A_{21}^{(k)} B_{11}^{(k)^{-1}} A_{12}^{(k)} \mathbf{v}_2^{(k)} \\
&\le \mathbf{v}_2^{(k)^T} (B^{(k-1)} - A^{(k-1)}) \mathbf{v}_2^{(k)} + \gamma^2 \mathbf{v}_2^{(k)^T} A^{(k-1)} \mathbf{v}_2^{(k)} \\
&\quad + b_1 \mathbf{v}_1^{(k)^T} A_{11}^{(k)} \mathbf{v}_1^{(k)} + \sigma_2 \gamma \zeta \mathbf{v}_1^{(k)^T} A_{11}^{(k)} \mathbf{v}_1^{(k)} + \gamma \zeta^{-1} \mathbf{v}_2^{(k)^T} A^{(k-1)} \mathbf{v}_2^{(k)} \\
&= (\gamma^2 + \gamma \zeta^{-1}) \sum_{s=1}^{k-1} \mathbf{v}^{(s)^T} A^{(s)} \mathbf{v}^{(s)} + (1 + \sigma_2 \gamma \zeta) b_1 \sum_{s=1}^{k} \mathbf{v}_1^{(s)^T} A_{11}^{(s)} \mathbf{v}_1^{(s)} \\
&\le C k^2 \mathbf{v}^T A^{(k)} \mathbf{v}.
\end{aligned}$$

We recall that $\sigma_2 \ge \lambda_{\max}[D_{11}^{(k)^{-1}} A_{11}^{(k)}]$ and $b_1 = \ell^2 \sigma_1$, where $\sigma_1 \ge \lambda_{\max}[A_{11}^{(k)^{-1}} D_{11}^{(k)}]$ and $\ell \ge \| D_{11}^{(k)^{-1/2}} L_{11}^{(k)^T} D_{11}^{(k)^{-1/2}} \|$. These constants ($\sigma_1$, σ_2, and ℓ) are mesh independent.

One can make some optimization with respect to $\zeta \in (0, \infty)$, but the result will still be of the same order, namely, $O(k^2)$. This bound is obtained based on estimates (9) and (10) with $\eta(\frac{h_s}{h_k}) = C(k - s) + 1$. □

4. Stabilization of the HB method, I: An algebraic approach—the AMLI method. Here we present the algebraic approach proposed in Axelsson and Vassilevski [5] for stabilizing the multilevel HB preconditioners. This stabilization is essential for 3-d problems. A similar approach was proposed in Kuznetsov [23]. That is, the method in [23] also exploits polynomially-based inner iterations (at all discretization levels) but is only applicable for certain finite difference problems.

Here we need polynomials $p_{\nu_k}^{(k)}(t)$ of degree ν_k at every discretization level k that are properly scaled such that, in the interval $(0, 1]$, the polynomials take values in

$[0, 1)$ and

$$p_{\nu_k}^{(k)}(0) = 1.$$

Some practical choices of $p_\nu(t)$ are specified after Definition 9.

We call the AMLI procedure as explained further in this section a stabilization of the HB method since all the HB multilevel methods from the previous sections are algebraically modified by introducing polynomially-based inner iterations (at certain discretization levels), exploiting recursive calls to coarse levels and based on the preconditioners defined by induction at those levels, in an optimal way. This does not change the nature of the HB methods—that all constants involved in various spectral relations can be estimated locally (with respect to the elements from the initial triangulation $\mathcal{T}_0$). Because of this, the AMLI methods preserve this locality property of the HB methods and, as a corollary, the resulting constants in the spectral equivalence relations are independent of possible large jumps in the coefficients of the bilinear form $A(.,.)$ as long as these only occur across element boundaries of elements from $\mathcal{T}_0$.

Also, the name *algebraic* does not necessarily refer to the algebraic generation of the coarse discretizations (and the respective coarse-level matrices), but is due to the polynomials involved in the definition of the multilevel iteration (or cycle). Thus, in this respect the AMLI methods are different from the algebraic multigrid methods as studied earlier in [34] and others.

On the other hand, the AMLI methods have much in common with the classical multigrid methods in the sense that the former are recursively defined from coarser to finer levels and involve recursive calls to coarser levels. An essential feature of the AMLI methods is that they allow for recursive calls not necessarily to all coarse discretization levels and still preserve their optimality property. Avoiding recursive calls at most discretization levels is important since it results in a less expensive operation count per preconditioning step. Details regarding the complexity of the method are found in section 4.6.

This large section is structured as follows.

- *AMLI methods that require certain parameters to estimate.* We discuss the minimum eigenvalues of ${M^{(k)}}^{-1}A^{(k)}$ at all discretization levels at which recursive calls to previous coarser levels exist. This eigenvalue estimation, as demonstrated in Vassilevski [38], can be performed adaptively from coarser to finer levels based on the Lanczos method. The AMLI methods here are natural extensions of the HB multilevel methods as studied in section 3 for both types of multiplicative schemes—the HBMG of Bank, Dupont, and Yserentant [10] (see Definition 6) generalized in Definition 9 below and the scheme of Vassilevski [37] (see Definition 4) generalized in Definition 7 below. We also consider a special version of AMLI methods that is based on (approximate) two-level Schur complements and which has further extensions to algebraically defined coarse-level matrices (i.e., not generated by successively refined meshes). This is the so-called Version I AMLI preconditioners as described in Definition 8. All these AMLI methods have (essentially one) additive version, and we present only a parameter-free variant (to avoid a priori estimation of whatever parameters) of additive AMLI methods in Definition 10 below.
- *Parameter-free AMLI methods.* The main idea here is to replace the polynomials involved in the recursive definition of the AMLI preconditioners by conjugate-gradient-type iterations. This, however, leads to nonlinear (and

possibly variable-step, i.e., changing from iteration to iteration) mappings, and therefore one needs to analyze such variable-step nonlinear preconditioned methods. This (additive) AMLI method is introduced in Definition 10 below.

4.1. The AMLI method. We first define the multiplicative or block Gauss–Seidel AMLI preconditioner.

DEFINITION 7 (the multiplicative or block Gauss–Seidel AMLI preconditioner (Axelsson and Vassilevski [5], [6] and Vassilevski [38])). *Set $M^{(0)} = A^{(0)}$. For $k \geq 1$ one defines*

$$M^{(k)} = \begin{bmatrix} B_{11}^{(k)} & 0 \\ A_{21}^{(k)} & \tilde{M}^{(k-1)} \end{bmatrix} \begin{bmatrix} I & {B_{11}^{(k)}}^{-1} A_{12}^{(k)} \\ 0 & I \end{bmatrix} \begin{matrix} \} & V_k^{(1)} \\ \} & V_{k-1} \end{matrix} .$$

Here

$$\tilde{M}^{(k-1)^{-1}} = \left[I - p_{\nu_{k-1}}^{(k-1)} \left(M^{(k-1)^{-1}} A^{(k-1)} \right) \right] A^{(k-1)^{-1}}. \tag{13}$$

It is clear that if $p \approx 0$ over the spectrum of $M^{(k-1)^{-1}} A^{(k-1)}$ then $\tilde{M}^{(k-1)} \approx A^{(k-1)}$; hence $M^{(k)}$ becomes close to a two-level preconditioner for $A^{(k)}$ of the form defined in Definition 1.

Note that the last expression (13) for $\tilde{M}^{(k-1)^{-1}}$ can be written so that it does not contain any inverses of $A^{(k-1)}$. Since $p_{\nu_{k-1}}^{(k-1)}(0) = 1$, $q(t) = \frac{1-p(t)}{t}$ (omitting the super- and subscripts of p) is also a polynomial. Hence

$$\tilde{M}^{(k-1)^{-1}} = q_{\nu_{k-1}}^{(k-1)} \left(M^{(k-1)^{-1}} A^{(k-1)} \right) M^{(k-1)^{-1}}.$$

However, $\tilde{M}^{(k-1)^{-1}}$ involves ν_{k-1} times the inverses of $M^{(k-1)}$, the preconditioner defined recursively on the previous discretization levels.

4.2. Version I AMLI preconditioners. There is one more version of the AMLI method (see Axelsson and Vassilevski [5]).

DEFINITION 8 (Version I AMLI preconditioners). *Let $B_{11}^{(k)} = {D_{11}^{(k)}}^{-1}$, for an explicitly given matrix $D_{11}^{(k)}$, be the given approximation to $A_{11}^{(k)}$ that satisfies the following spectral equivalence inequalities:*

$$\mathbf{v}_1^T A_{11}^{(k)} \mathbf{v}_1 \leq \mathbf{v}_1^T B_{11}^{(k)} \mathbf{v}_1 \leq (1 + b_1) \mathbf{v}_1^T A_{11}^{(k)} \mathbf{v}_1 \quad \textit{for all } \mathbf{v}_1.$$

As before, the constant $b_1 \geq 0$ is assumed mesh (or level) independent.

One then defines the approximate Schur complements $S_D^{(k)}$ whose actions on vectors are inexpensively available:

$$S_D^{(k)} = A_{22}^{(k)} - A_{21}^{(k)} D_{11}^{(k)} A_{12}^{(k)}.$$

Then, letting $B^{(0)} = A^{(0)}$ for $k = 1, 2, \ldots$, one proceeds as follows:

$$B^{(k)} = \begin{bmatrix} {D_{11}^{(k)}}^{-1} & 0 \\ A_{21}^{(k)} & \tilde{S}^{(k)} \end{bmatrix} \begin{bmatrix} I & D_{11}^{(k)} A_{12}^{(k)} \\ 0 & I \end{bmatrix}.$$

Here

$$\tilde{S}^{(k)^{-1}} = \left[I - p_\nu \left(B^{(k-1)^{-1}} S_D^{(k)} \right) \right] S_D^{(k)^{-1}}.$$

The polynomial $p_\nu = p^{(k-1)}_{\nu_{k-1}}$ *is properly scaled such that* $p_\nu(0) = 1$ *and* p_ν *takes values in* $[0, 1)$ *for* $t \in (0, 1]$.

We first remark that $q_{\nu-1} = \frac{1-p_\nu(t)}{t}$ is also a polynomial (since $p_\nu(0) = 1$) and hence

$$\tilde{S}^{(k)^{-1}} = q_{\nu-1}\left(B^{(k-1)^{-1}} S_D^{(k)}\right) B^{(k-1)^{-1}}.$$

This shows that to compute the inverse actions of $\tilde{S}^{(k)}$ one must solve ν_{k-1} systems with $B^{(k-1)}$, which is of a factored form (but involves possible recursive calls to previous coarse levels).

It is clear that we may not have the coarse-level matrices $A^{(k)}$ available at all. Then Definition 8 also works when the coarse-level matrix $A^{(k-1)}$ is defined from $A^{(k)}$ by letting $A^{(k-1)} = S_D^{(k)}$, that is, for algebraically generated coarse-level matrices. This algebraic generation is computationally feasible if $D_{11}^{(k)}$ is sparse (e.g., diagonal) and if the blocks $A_{12}^{(k)}$ and $A_{21}^{(k)}$ have a simple structure, such that the product $A_{21}^{(k)} D_{11}^{(k)} A_{12}^{(k)}$ does not increase the fill-in too much. Then what is left is to define (e.g., based on the matrix graph) a two-by-two block structure of any successive coarse matrix $A^{(k-1)}$. For more detail we refer to Axelsson and Neytcheva [3].

4.3. Spectral equivalence properties of the AMLI methods. For practical purposes, one lets $\nu_k = 1$ at most of the levels; i.e., there is no recursion involved at most of the levels. Also, as recently demonstrated by Axelsson and Neytcheva [4] and Neytcheva [30], [31], one should also choose the coarse discretization sufficiently fine in order to be able to efficiently implement the method, including on some massively parallel machines such as CM–200.

The method is of optimal order if proper relation holds between the polynomial degree ν and the number of consecutive levels k_0 at which we do not nest the algorithm (see relation (20)). This means that only at the levels with index k of multiplicity k_0 (i.e., $k = sk_0,\ s = 1, 2, \ldots$) do we use polynomials of degree $\nu > 1$. Originally, the AMLI method as proposed in Axelsson and Vassilevski [5], [6] corresponded to the case $k_0 = 1$ which imposed a certain restriction on the constant γ in the strengthened Cauchy inequality (or, equivalently, on the constant η_1) in the sense that the method has an optimal complexity in this case if $\sqrt{\eta_1} = \sqrt{\frac{1}{1-\gamma^2}} < \nu < 2^d$ ($d = 2$ for 2-d domain Ω and $d = 3$ for Ω a 3-d polytope). This shows that the AMLI method (for $k_0 = 1$) will be at least as expensive as a W-cycle multigrid method; i.e., $\nu_k \geq \nu \geq 2$. The general case $k_0 \geq 1$ was considered and analyzed in Vassilevski [38], where the optimality of the method from Definition 7 was proven for finite element second-order elliptic bilinear forms (1), in general, for k_0 sufficiently large and $\nu_{sk_0} = \nu$, $s = 1, 2, \ldots, [\frac{J}{k_0}]$ properly chosen (such as in (20)). This choice $k_0 \geq 1$ relaxes the complexity of the corresponding AMLI methods (since in this case we do not have to nest the method at all discretization levels).

For the Version I AMLI preconditioner from Definition 8 a similar result holds.

THEOREM 5. *Let* $B_{11}^{(k)} = D_{11}^{(k)^{-1}}$ *(it is commonly assumed that* $D_{11}^{(k)}$ *is given explicitly) be a symmetric positive-definite approximation to* $A_{11}^{(k)}$ *that satisfies the uniform spectral equivalence estimates for a mesh- (or level-) independent constant* $b_1 \geq 0$*:*

$$\mathbf{v}_1^T A_{11}^{(k)} \mathbf{v}_1 \leq \mathbf{v}_1^T B_{11}^{(k)} \mathbf{v}_1 \leq (1 + b_1) \mathbf{v}_1^T A_{11}^{(k)} \mathbf{v}_1 \quad \textit{for all } \mathbf{v}_1.$$

Given an integer $k_0 \geq 1$*, let* $\nu_{k-1} = 1$ *for* $k-1 \neq (s-1)k_0$ *in Definition* 8*. Choose* $\nu > \sqrt{\eta_1 \eta_{k_0-1} \frac{1+b_1\eta_1}{1+b_1}}$*, where* $\eta_{k_0-1} = \eta(\frac{h_m}{h_{m+k_0-1}})$*,* $\eta(.)$ *is defined as in* (10)*, and* $\eta_1 = \frac{1}{1-\gamma^2}$*. The constant* $b_1 \geq 0$ *takes part in the spectral equivalence relations between* $A_{11}^{(k)}$ *and* $B_{11}^{(k)}$*. We write shortly* $\eta_r = \eta(\frac{h_s}{h_{s+r}})$ *(noting that the expression is independent of* $s \geq 0$*). Also, let* $\alpha \in (0,1)$ *be sufficiently small such that the following inequality holds:*

$$\frac{1+b_1\eta_1}{1+b_1}\eta_1 \frac{(1-\tilde{\alpha})^\nu}{\alpha \left[\sum\limits_{r=1}^{\nu}(1+\sqrt{\tilde{\alpha}})^{\nu-r}(1-\sqrt{\tilde{\alpha}})^{r-1}\right]^2}$$

$$\leq \frac{1}{\eta_{k_0-1}}\left[\frac{1}{\alpha} - \left(1 + (1+b_1\eta_1)\sum_{s=1}^{k_0}\eta_s\right)\right] \quad \left(\tilde{\alpha} = (1-\gamma^2)\,\alpha = \frac{\alpha}{\eta_1}\right). \tag{14}$$

Such a sufficiently small α *exists since for* $\alpha \to 0$ *(after multiplying* (14) *by* α*) we have* $\eta_1 \frac{1+b_1\eta_1}{1+b_1}\frac{1}{\nu^2} < \frac{1}{\eta_{k_0-1}}$ *(which has already been assumed). Consider then the Version* I *AMLI preconditioner* $B^{(k)}$ *from Definition* 8 *for polynomials*

$$p^{(k-1)}_{\nu_{k-1}}(t) = \frac{1+T_\nu\left(\frac{1+\tilde{\alpha}-2t}{1-\tilde{\alpha}}\right)}{1+T_\nu\left(\frac{1+\tilde{\alpha}}{1-\tilde{\alpha}}\right)},$$

with $\nu_{k-1} = \nu$ *and* $k-1 = (s-1)k_0$*,* $s = 1, 2, \ldots, [\frac{J}{k_0}]$ *(the integer part of* $\frac{J}{k_0}$*) and* $p^{(k-1)}_{\nu_{k-1}} = 1-t$ *for all remaining* k*; i.e.,* $\nu_{k-1} = 1$ *for* $k-1 \neq (s-1)k_0$*. Here* T_ν *is the Chebyshev polynomial of the first kind of degree* ν*.*

Then the following spectral equivalence relations hold:

$$\mathbf{v}^T A^{(k)}\mathbf{v} \leq \mathbf{v}^T B^{(k)}\mathbf{v} \leq \frac{1}{\alpha}\mathbf{v}^T A^{(k)}\mathbf{v} \quad \textit{for all } \mathbf{v}.$$

Note that if $b_1 = 0$, i.e., $B_{11}^{(k)} = A_{11}^{(k)}$, which means that one uses the exact Schur complements $S^{(k)}$ in Definition 8, the assumption on ν and k_0 reads $\nu > \sqrt{\eta_1\eta_{k_0-1}}$. In the simplest case $k_0 = 1$ the relation reads $\nu > \frac{1}{\sqrt{1-\gamma^2}}$, already shown in Axelsson and Vassilevski [5]. For the general estimate, letting $b_1 \to \infty$, one gets the worst-case relation between ν and k_0; namely, $\nu > \eta_1\sqrt{\eta_{k_0-1}}$.

Proof. Given $m = (s-1)k_0$, consider any k, $m < k \leq \min(sk_0, J)$. We have, noting that $\tilde{S}^{(l)} = B^{(l-1)}$ for $m+1 < l \leq k$,

$$\begin{aligned}\mathbf{v}^T(B^{(k)} - A^{(k)})\mathbf{v} = &\sum_{l=m+1}^{k} \mathbf{v}_1^{(l)^T}(B_{11}^{(l)} - A_{11}^{(l)})\mathbf{v}_1^{(l)} \\ &+ \mathbf{v}^{(m)^T}\left(\tilde{S}^{(m+1)} - S_D^{(m+1)}\right)\mathbf{v}^{(m)} \\ &+ \sum_{l=m+1}^{k} \mathbf{v}^{(l-1)^T} A_{21}^{(l)} B_{11}^{(l)^{-1}} A_{12}^{(l)}\mathbf{v}^{(l-1)}.\end{aligned} \tag{15}$$

The notation of the vectors $\mathbf{v}^{(l)}$ and $\mathbf{v}_1^{(l)}$ used in (15) is the same as in the proof of Theorem 3.

We first see that expression (15) implies the positive semidefiniteness of $B^{(k)}-A^{(k)}$ since all terms in (15) are positive semidefinite. For the term containing $\tilde{S}^{(m+1)}-S_D^{(m+1)}$ this follows from the definition of $\tilde{S}^{(m+1)}$ and the choice of p_ν. The upper bound of the spectrum of ${A^{(k)}}^{-1}B^{(k)}$ is obtained by induction as follows. Assume (by induction) that $\lambda[{A^{((s-1)k_0)}}^{-1}B^{((s-1)k_0)}]\in[1,1+\delta_s]$, where

$$\alpha\le\frac{1}{1+\delta_s}. \tag{16}$$

We next estimate the spectrum of ${A^{(sk_0)}}^{-1}B^{(sk_0)}$. Note first that $A^{(m)}-S_D^{(m+1)}=A_{21}^{(m+1)}D_{11}^{(m+1)}A_{12}^{(m+1)}$, which shows the inequality

$$\mathbf{v}^TB^{(m)}\mathbf{v}\ge\mathbf{v}^TA^{(m)}\mathbf{v}\ge\mathbf{v}^TS_D^{(m+1)}\mathbf{v}. \tag{17}$$

Therefore, $\lambda[{B^{(m)}}^{-1}S_D^{(m+1)}]\in(0,1]$. Next, one has the inequality

$$\begin{aligned}\mathbf{v}^TA^{(m)}\mathbf{v}&\le\eta_1\inf_{\mathbf{w}_1}\begin{bmatrix}\mathbf{w}_1\\ \mathbf{v}\end{bmatrix}^TA^{(m+1)}\begin{bmatrix}\mathbf{w}_1\\ \mathbf{v}\end{bmatrix}\\&=\eta_1\mathbf{v}^T\left(A^{(m)}-A_{21}^{(m+1)}{A_{11}^{(m+1)}}^{-1}A_{12}^{(m+1)}\right)\mathbf{v}.\end{aligned}$$

This inequality, with $\eta_1=\frac{1}{1-\gamma^2}$, implies

$$\mathbf{v}^TA_{21}^{(m+1)}{A_{11}^{(m+1)}}^{-1}A_{12}^{(m+1)}\mathbf{v}\le\gamma^2\mathbf{v}^TA^{(m)}\mathbf{v},$$

which in turn shows

$$\begin{aligned}\mathbf{v}^T(A^{(m)}-S_D^{(m+1)})\mathbf{v}&\le\mathbf{v}^TA_{21}^{(m+1)}{A_{11}^{(m+1)}}^{-1}A_{12}^{(m+1)}\mathbf{v}\\&\le\gamma^2\mathbf{v}^TA^{(m)}\mathbf{v}.\end{aligned}$$

One then obtains

$$\mathbf{v}^TS_D^{(m+1)}\mathbf{v}\ge(1-\gamma^2)\mathbf{v}^TA^{(m)}\mathbf{v}.$$

This inequality and (16) imply the estimate

$$\lambda_{\min}\left[{B^{(m)}}^{-1}S_D^{(m+1)}\right]\ge\lambda_{\min}\left[{B^{(m)}}^{-1}A^{(m)}\right](1-\gamma^2)\ge\frac{1-\gamma^2}{1+\delta_s}.$$

The latter inequality and estimate (17) show that the spectrum of ${B^{(m)}}^{-1}S_D^{(m+1)}$ is contained in $[\frac{1-\gamma^2}{1+\delta_s},1]$.

Therefore, we get the following estimate:

$$\lambda\left[\left(S_D^{((s-1)k_0+1)}\right)^{-1}\tilde{S}^{((s-1)k_0+1)}\right]\in[1,1+\tilde{\delta}_s],$$

where

$$\begin{aligned}\tilde{\delta}_s&\le\sup\left\{\frac{1}{1-p_\nu(t)}-1,\ t\in\left[\frac{1-\gamma^2}{1+\delta_s},1\right]\right\}\\&\le\sup\left\{\frac{1}{1-p_\nu(t)}-1,\ t\in[\alpha(1-\gamma^2),1]\right\}\\&=\sup\left\{\frac{p_\nu(t)}{1-p_\nu(t)},\ t\in[\tilde{\alpha},1]\right\}.\end{aligned}$$

Here we have used the fact that $[\frac{1}{1+\delta_s}, 1] \subset [\alpha, 1]$ (see (16)). Since

$$\sup_{t\in[\tilde{\alpha},1]} \left| T_\nu \left(\frac{1+\tilde{\alpha}-2t}{1-\tilde{\alpha}} \right) \right| = 1,$$

we obtain

$$\begin{aligned}\sup\{p_\nu(t),\ t\in[\tilde{\alpha},1]\} &= \frac{2}{1+T_\nu\left(\frac{1+\tilde{\alpha}}{1-\tilde{\alpha}}\right)} \\ &= \frac{2}{1+\frac{1+q^{2\nu}}{2q^\nu}}, \quad q = \frac{1-\sqrt{\tilde{\alpha}}}{1+\sqrt{\tilde{\alpha}}}.\end{aligned}$$

Hence,

$$\begin{aligned}\tilde{\delta}_s &\le \frac{2}{T_\nu\left(\frac{1+\tilde{\alpha}}{1-\tilde{\alpha}}\right)-1} = \frac{4q^\nu}{(q^\nu-1)^2} \\ &= \frac{(1-\tilde{\alpha})^\nu}{\tilde{\alpha}\left[\sum_{l=1}^{\nu}(1+\sqrt{\tilde{\alpha}})^{\nu-l}(1-\sqrt{\tilde{\alpha}})^{l-1}\right]^2}.\end{aligned} \tag{18}$$

Now using (15) and (4) with $\zeta=\gamma$ and the fact that $\eta_1 = \frac{1}{1-\gamma^2}$, one gets

$$\begin{aligned}\mathbf{v}^T(B^{(k)}-A^{(k)})\mathbf{v} \le b_1 &\sum_{l=m+1}^{k} \mathbf{v}_1^{(l)^T} A_{11}^{(l)} \mathbf{v}_1^{(l)} + \tilde{\delta}_s \mathbf{v}^{(m)^T} S_D^{(m+1)} \mathbf{v}^{(m)} \\ &+ \sum_{l=m+1}^{k} \mathbf{v}^{(l-1)^T} A_{21}^{(l)} A_{11}^{(l)^{-1}} A_{12}^{(l)} \mathbf{v}^{(l-1)} \\ \le b_1\eta_1 &\sum_{l=m+1}^{k} \mathbf{v}^{(l)^T} A^{(l)} \mathbf{v}^{(l)} + \sum_{l=m+1}^{k} \mathbf{v}^{(l-1)^T} A^{(l-1)} \mathbf{v}^{(l-1)} \\ &+ \tilde{\delta}_s \mathbf{v}^{(m)^T} S_D^{(m+1)} \mathbf{v}^{(m)}.\end{aligned}$$

We next need the following inequality, which is proved based on the spectral equivalence relation between $A_{11}^{(m+1)}$ and $B_{11}^{(m+1)}$, the fact that $S^{(m+1)}$ is a Schur complement of $A^{(m+1)}$, and the definition of $\eta_1 = \eta(\frac{h_m}{h_{m+1}})$:

$$\begin{aligned}\mathbf{v}^{(m)^T} S_D^{(m+1)} \mathbf{v}^{(m)} &= \mathbf{v}^{(m)^T}(A^{(m)} - A_{21}^{(m+1)} B_{11}^{(m+1)^{-1}} A_{12}^{(m+1)})\mathbf{v}^{(m)} \\ &\le \mathbf{v}^{(m)^T} A^{(m)} \mathbf{v}^{(m)} - \frac{1}{b_1+1} \mathbf{v}^{(m)^T} A_{21}^{(m+1)} A_{11}^{(m+1)^{-1}} A_{12}^{(m+1)} \mathbf{v}^{(m)} \\ &= \frac{b_1}{1+b_1} \mathbf{v}^{(m)^T} A^{(m)} \mathbf{v}^{(m)} + \frac{1}{1+b_1} \mathbf{v}^{(m)^T} S^{(m+1)} \mathbf{v}^{(m)} \\ &\le \frac{1+b_1\eta_1}{1+b_1} \mathbf{v}^{(m+1)^T} A^{(m+1)} \mathbf{v}^{(m+1)}.\end{aligned}$$

The last two inequalities (for $\mathbf{v}^T(B^{(k)} - A^{(k)})\mathbf{v}$ and $\mathbf{v}^{(m)^T} S_D^{(m+1)} \mathbf{v}^{(m)}$) and the definition of $\eta_l = \eta(\frac{h_s}{h_{s+l}})$ imply

$$\begin{aligned}
\mathbf{v}^T(B^{(k)} - A^{(k)})\mathbf{v} &\leq b_1\eta_1 \sum_{l=m+1}^{k} \mathbf{v}^{(l)^T} A^{(l)} \mathbf{v}^{(l)} + \sum_{l=m+1}^{k} \mathbf{v}^{(l-1)^T} A^{(l-1)} \mathbf{v}^{(l-1)} \\
&\quad + \tilde{\delta}_s \, \frac{1 + b_1\eta_1}{1 + b_1} \, \mathbf{v}^{(m+1)^T} A^{(m+1)} \mathbf{v}^{(m+1)} \\
&\leq \left[(1 + b_1\eta_1) \sum_{l=m}^{k-1} \eta_{k-l} + \tilde{\delta}_s \, \frac{1 + b_1\eta_1}{1 + b_1} \, \eta_{k-m-1} \right] \mathbf{v}^{(k)^T} A^{(k)} \mathbf{v}^{(k)} \\
&\leq \left[(1 + b_1\eta_1) \sum_{l=1}^{k_0} \eta_l + \tilde{\delta}_s \frac{1 + b_1\eta_1}{1 + b_1} \, \eta_{k_0-1} \right] \mathbf{v}^T A^{(k)} \mathbf{v} \\
&\leq \left[(1 + b_1\eta_1) \sum_{l=1}^{k_0} \eta_l + \eta_{k_0-1} \frac{1 + b_1\eta_1}{1 + b_1} \, \frac{(1 - \tilde{\alpha})^\nu}{\tilde{\alpha} \left[\sum\limits_{l=1}^{\nu} (1 + \sqrt{\tilde{\alpha}})^{\nu - l} (1 - \sqrt{\tilde{\alpha}})^{l-1} \right]^2} \right] \\
&\quad \times \mathbf{v}^T A^{(k)} \mathbf{v} \\
&\leq \left(\frac{1}{\alpha} - 1 \right) \mathbf{v}^T A^{(k)} \mathbf{v}.
\end{aligned}$$

The last inequality is obtained using (18) and (14).

Therefore, we have established that

$$1 + \delta_{s+1} \leq \frac{1}{\alpha} \text{ or } \alpha \leq \frac{1}{1 + \delta_{s+1}},$$

which confirms the induction assumption (16) for $s := s + 1$. ∎

4.4. The HBMG–AMLI method. The HBMG preconditioner from Definition 6 can be similarly stabilized. For the case $k_0 = 1$ the above polynomial-type stabilization of the HBMG method was exploited by Guo [21] (although in this case ($k_0 = 1$) the proof in [21] of the complexity of the method was not actually as satisfactory). Here we consider the more general case $k_0 \geq 1$, which is more practical since it does not require nesting the algorithm at all discretization levels and still is able to achieve both optimal relative condition number and optimal complexity of the corresponding AMLI preconditioners.

Definition 9 (multiplicative or block Gauss–Seidel HBMG–AMLI preconditioning scheme). *Assume that $A_{11}^{(k)}$ is split as*

$$A_{11}^{(k)} = D_{11}^{(k)} + L_{11}^{(k)} + L_{11}^{(k)^T},$$

where $L_{11}^{(k)}$ is a strictly lower triangular part of $A_{11}^{(k)}$ and $D_{11}^{(k)}$ is a simple part of $A_{11}^{(k)}$; i.e., $D_{11}^{(k)}$ is an easy-to-factor or to-solve-systems matrix (e.g., the scalar diagonal part of $A_{11}^{(k)}$). It is also assumed that $D_{11}^{(k)}$ is symmetric and positive definite.

Define $B^{(0)} = A^{(0)}$. For $k \geq 1$ assume by induction that $B^{(k-1)}$, the HBMG–AMLI preconditioner for $A^{(k-1)}$, has been defined. Then

$$B^{(k)} = \begin{bmatrix} L_{11}^{(k)} + D_{11}^{(k)} & 0 \\ A_{21}^{(k)} & I \end{bmatrix} \begin{bmatrix} D_{11}^{(k)^{-1}} & 0 \\ 0 & \tilde{B}^{(k-1)} \end{bmatrix} \begin{bmatrix} L_{11}^{(k)^T} + D_{11}^{(k)} & A_{12}^{(k)} \\ 0 & I \end{bmatrix} \begin{array}{l} \} \quad V_k^{(1)} \\ \} \quad V_{k-1} \end{array}.$$

Here

$$\tilde{B}^{(k-1)^{-1}} = \left[I - p_{\nu_{k-1}}^{(k-1)}\left(B^{(k-1)^{-1}} A^{(k-1)}\right)\right] A^{(k-1)^{-1}}.$$

The polynomials $p_{\nu_k}^{(k)}$ are as in Definition 5; i.e., $p_{\nu_k}^{(k)}$ are properly scaled such that, in the interval $(0, 1]$, the polynomials take values in $[0, 1)$ and

$$p_{\nu_k}^{(k)}(0) = 1.$$

For practical purposes $\nu_k = 1$ at most of the levels k. A simple choice is $p_\nu(t) = (1-t)^\nu$, while a more complicated one is

$$p_\nu(t) = \frac{1 + T_\nu\left(\frac{1+\alpha-2t}{1-\alpha}\right)}{1 + T_\nu\left(\frac{1+\alpha}{1-\alpha}\right)},$$

where $\alpha \in (0, 1]$ is such that $\alpha \leq \lambda_{\min}[B^{(k)^{-1}} A^{(k)}]$. Here T_ν is the Chebyshev polynomial of the first kind of degree ν. The last choice of $p_\nu(t)$ requires estimates of the parameter $\alpha = \alpha_k$ (i.e., of the minimum eigenvalue of $B^{(k)^{-1}} A^{(k)}$). As was demonstrated in Vassilevski [38], this can be done adaptively. Alternatively, one could use inner iterations by a conjugate-gradient-type iteration method with a variable-step preconditioner (i.e., a nonlinear preconditioner). In this way one ends up with a variable-step AMLI preconditioner which is a nonlinear mapping. This preconditioner was introduced and analyzed in Axelsson and Vassilevski [8] and is defined below (see Definition 10).

To analyze the HBMG–AMLI method (using the same notation as introduced in the proof of Theorem 5, i.e., letting $m = (s-1)k_0$ and $k : m < k \leq \min(sk_0, J)$), a starting point is an identity similar to (12) and the inequalities which it implies. We have, for any $\zeta > 0$,

$$\begin{aligned} \mathbf{v}^T(B^{(k)} - A^{(k)})\mathbf{v} \leq{}& \mathbf{v}_2^{(k)^T}(\tilde{B}^{(k-1)} - A^{(k-1)})\mathbf{v}_2^{(k)} + \mathbf{v}_1^{(k)^T} L_{11}^{(k)} D_{11}^{(k)^{-1}} L_{11}^{(k)^T} \mathbf{v}_1^{(k)} \\ &+ 2\mathbf{v}_1^{(k)^T} L_{11}^{(k)} D_{11}^{(k)^{-1}} A_{12}^{(k)} \mathbf{v}_2^{(k)} + \mathbf{v}_2^{(k)^T} A_{21}^{(k)} B_{11}^{(k)^{-1}} A_{12}^{(k)} \mathbf{v}_2^{(k)} \\ \leq{}& \mathbf{v}_2^{(k)^T}(\tilde{B}^{(k-1)} - A^{(k-1)})\mathbf{v}_2^{(k)} + \gamma^2 \mathbf{v}_2^{(k)^T} A^{(k-1)} \mathbf{v}_2^{(k)} \\ &+ b_1 \mathbf{v}_1^{(k)^T} A_{11}^{(k)} \mathbf{v}_1^{(k)} + \sigma_2\gamma\zeta \mathbf{v}_1^{(k)^T} A_{11}^{(k)} \mathbf{v}_1^{(k)} + \gamma\zeta^{-1} \mathbf{v}_2^{(k)^T} A^{(k-1)} \mathbf{v}_2^{(k)}. \end{aligned}$$

The latter inequality, used recursively, implies

(19)

$$\begin{aligned} \mathbf{v}^T(B^{(k)} - A^{(k)})\mathbf{v} \leq{}& (\gamma^2 + \gamma\zeta^{-1}) \sum_{l=m+1}^{k-1} \mathbf{v}^{(l)^T} A^{(l)} \mathbf{v}^{(l)} \\ &+ (1 + \sigma_2\gamma\zeta) b_1 \sum_{l=m+1}^{k} \mathbf{v}_1^{(l)^T} A_{11}^{(l)} \mathbf{v}_1^{(l)} + \mathbf{v}^{(m)^T}(\tilde{B}^{(m)} - A^{(m)})\mathbf{v}^{(m)}. \end{aligned}$$

We recall that $\sigma_2 \geq \lambda_{\max}[D_{11}^{(k)^{-1}} A_{11}^{(k)}]$ and $b_1 = \ell^2\sigma_1$, where $\sigma_1 \geq \lambda_{\max}[A_{11}^{(k)^{-1}} D_{11}^{(k)}]$ and $\ell \geq \|D_{11}^{(k)^{-1/2}} L_{11}^{(k)^T} D_{11}^{(k)^{-1/2}}\|$. These constants ($\sigma_1$, σ_2, and ℓ) are mesh independent.

The term

$$\mathbf{v}^{(m)^T}(\tilde{B}^{(m)} - A^{(m)})\mathbf{v}^{(m)}$$

is estimated similarly as in the proof of Theorem 5. One gets, assuming (16) (where δ_s is such that $\lambda[A^{((s-1)k_0)^{-1}} B^{((s-1)k_0)}] \in [1, 1+\delta_s]$), that

$$\lambda\left[A^{((s-1)k_0)^{-1}} \tilde{B}^{((s-1)k_0)}\right] \in [1, 1+\tilde{\delta}_s],$$

where

$$\begin{aligned}
\tilde{\delta}_s &\leq \sup\left\{\frac{1}{1-p_\nu(t)} - 1,\ t \in \left[\frac{1}{1+\delta_s}, 1\right]\right\} \\
&\leq \sup\left\{\frac{1}{1-p_\nu(t)} - 1,\ t \in [\alpha, 1]\right\} \\
&= \sup\left\{\frac{p_\nu(t)}{1-p_\nu(t)},\ t \in [\alpha, 1]\right\}.
\end{aligned}$$

In the same way as in the proof of Theorem 5, one then proves (18). Then (19) together with (18), the definition of η_l (introduced in the formulation of Theorem 5), and inequality (4) used for $\zeta = \gamma$ leads us to

$$\begin{aligned}
\mathbf{v}^T(B^{(k)} - A^{(k)})\mathbf{v} \leq (\gamma^2 + \gamma\zeta^{-1}) &\sum_{l=m+1}^{k-1} \mathbf{v}^{(l)^T} A^{(l)} \mathbf{v}^{(l)} \\
&+ (1 + \sigma_2\gamma\zeta) b_1 \eta_1 \sum_{l=m+1}^{k} \mathbf{v}^{(l)^T} A^{(l)} \mathbf{v}^{(l)} + \tilde{\delta}_s \mathbf{v}^{(m)^T} A^{(m)} \mathbf{v}^{(m)} \\
\leq \eta_{k_0} &\frac{(1-\alpha)^\nu}{\alpha\left[\sum\limits_{l=1}^{\nu}(1+\sqrt{\alpha})^{\nu-l}(1-\sqrt{\alpha})^{l-1}\right]^2} \mathbf{v}^T A^{(k)} \mathbf{v} \\
&+ \left[\gamma^2 + \gamma\zeta^{-1} + (1+\sigma_2\gamma\zeta) b_1 \eta_1\right] \left(\sum_{l=1}^{k_0} \eta_l\right) \mathbf{v}^T A^{(k)} \mathbf{v} \\
\leq &\left(\frac{1}{\alpha} - 1\right) \mathbf{v}^T A^{(k)} \mathbf{v}.
\end{aligned}$$

The last inequality holds for sufficiently small $\alpha \in (0, 1]$ if $\nu > \sqrt{\eta_{k_0}}$. Therefore, we have proven the following theorem.

THEOREM 6. *The HBMG–AMLI method from Definition* 9 *gives spectrally equivalent preconditioners to $A^{(k)}$ provided p_ν are chosen as properly scaled and shifted Chebyshev polynomials with $\nu > \sqrt{\eta_{k_0}}$, and this is only at the levels with indices of multiplicity k_0. More precisely, let $\alpha \in (0, 1]$ be sufficiently small such that*

$$\eta_{k_0} \frac{(1-\alpha)^\nu}{\alpha\left[\sum\limits_{l=1}^{\nu}(1+\sqrt{\alpha})^{\nu-l}(1-\sqrt{\alpha})^{l-1}\right]^2} + 1 + \left[\gamma^2 + \gamma\zeta^{-1} + (1+\sigma_2\gamma\zeta) b_1 \eta_1\right] \left(\sum_{l=1}^{k_0} \eta_l\right) \leq \frac{1}{\alpha}.$$

Here ζ *is any fixed positive parameter and* $\gamma = \sqrt{1 - \frac{1}{\eta_1}}$. *Then the following spectral equivalence relations hold:*

$$\mathbf{v}^T A^{(k)} \mathbf{v} \le \mathbf{v}^T B^{(k)} \mathbf{v} \le \frac{1}{\alpha} \mathbf{v}^T A^{(k)} \mathbf{v} \quad \textit{for all } \mathbf{v}.$$

4.5. Variable-step AMLI methods. To introduce the variable-step AMLI method from Axelsson and Vassilevski [8] we first define a variable-step preconditioned conjugate-gradient method for solving the system

$$A\mathbf{x} = \mathbf{b}.$$

Here A is a given symmetric positive-definite matrix. Let $B[.]$ be a given, generally nonlinear, mapping that satisfies the following estimates.

- Coercivity estimate:

$$\mathbf{v}^T B[\mathbf{v}] \ge \delta_1 \mathbf{v}^T A^{-1} \mathbf{v}$$

for some positive constant δ_1.

- Boundedness estimate:

$$(B[\mathbf{v}])^T A B[\mathbf{v}] \le \delta_2^2 \mathbf{v}^T A^{-1} \mathbf{v}$$

for some positive constant δ_2.

ALGORITHM (variable-step conjugate-gradient method).

(0) initiate

$$\begin{aligned}
\mathbf{x} &= \mathbf{x}_0 \text{ −initial iterate;}\\
\mathbf{r} &= \mathbf{r}_0 = \mathbf{b} - A\mathbf{x}_0 \text{ – initial residual;}\\
\mathbf{d} &= \mathbf{d}_0 = B[\mathbf{r}_0] \text{ – initial search direction;}
\end{aligned}$$

(i) **For** $i = 0, 1, \ldots, \nu$ **compute**

$$\begin{aligned}
\mathbf{g} &= A\mathbf{d};\\
\gamma &= \mathbf{d}^T\mathbf{g};\\
\alpha &= \tfrac{1}{\gamma}\mathbf{r}^T\mathbf{d};\\
\mathbf{x} &= \mathbf{x} + \alpha\mathbf{d};\\
\mathbf{r} &= \mathbf{r} - \alpha\mathbf{g};\\
\tilde{\mathbf{r}} &= B[\mathbf{r}];\\
\mathbf{g} &= A\tilde{\mathbf{r}};\\
\beta &= \tfrac{1}{\gamma}\mathbf{g}^T\mathbf{d};\\
\mathbf{d} &= \tilde{\mathbf{r}} - \beta\mathbf{d};
\end{aligned}$$

(ii) **End**. □

It is not as hard to show (see, e.g., Axelsson and Vassilevski [7]) the following steepest descent rate of convergence:

$$\|\mathbf{b} - A\mathbf{x}_\nu\|_{A^{-1}} \le \left(\sqrt{1 - \left(\frac{\delta_1}{\delta_2}\right)^2}\right)^\nu \|\mathbf{b}\|_{A^{-1}}.$$

Here $\mathbf{x}_i$ is the ith iterate and we have assumed $\mathbf{x}_0 = 0$.

We are now in a position to define the variable-step AMLI preconditioner.

DEFINITION 10 (variable-step AMLI preconditioner (Axelsson and Vassilevski [8])). *Given an integer parameter $k_0 \geq 1$, use the block partitioning as in Definition* 5 *to define*

$$M^{(k_0)} = \begin{bmatrix} B_{11}^{(k_0)^{-1}} & 0 & & 0 \\ 0 & \ddots & \ddots & \\ & \ddots & B_{11}^{(1)^{-1}} & 0 \\ 0 & & 0 & A^{(0)^{-1}} \end{bmatrix}.$$

For $m = (s-1)k_0 + 1, \ldots, \min(J, sk_0)$, $k = (s-1)k_0$, and $s = 2, 3, \ldots$ one further defines

$$M^{(m)}[\cdot] = \begin{bmatrix} B_{11}^{(m)^{-1}} & 0 & & & 0 \\ 0 & B_{11}^{(m-1)^{-1}} & 0 & & \\ & \ddots & \ddots & \ddots & \\ & & 0 & B_{11}^{(k+1)^{-1}} & 0 \\ 0 & & & 0 & \tilde{M}_\nu^{(k)}[.] \end{bmatrix},$$

where $\tilde{M}_\nu^{(k)}[\mathbf{b}]$, for any given $\mathbf{b}$, is defined by applying ν steps of the algorithm to solve the system

$$A^{(k)}\mathbf{x} = \mathbf{b},$$

using $M^{(k)}[.]$ (already defined at the previous coarse levels by recursion) as a variable-step preconditioner and $\mathbf{x}_0 = 0$ as an initial iterate. Then $\tilde{M}_\nu^{(k)}[\mathbf{b}] = \mathbf{x}_\nu$, the νth iterate.

The method was analyzed in Axelsson and Vassilevski [8] and the following result proven.

THEOREM 7. *Assume that ν, the number of inner variable-step preconditioned conjugate-gradient iterations, is sufficiently large such that for any given fixed $\epsilon \in (0,1)$,*

$$\nu \geq \frac{\log \epsilon^2}{\log \left[1 - \left(\frac{1-\epsilon}{1+\epsilon}\right)^2 (CH_{k_0})^{-2}\right]} = O(H_{k_0}^2)\left(\frac{1+\epsilon}{1-\epsilon}\right)^2 \log \epsilon^{-2}, \quad k_0 \to \infty.$$

Here C is a constant coming from the strengthened Cauchy inequality (11) *and $H_{k_0} = \eta_1 \sum_{l=1}^{k_0} \eta_l + \eta_{k_0}$. The constants $\{\eta_l\}$ are introduced in Theorem* 5. *In other words, let ν be sufficiently large such that (we assume here that $\frac{h_s}{h_{s+1}} = 2$; h_s is the mesh size at the sth discretization level)*

$$\nu \geq \begin{cases} Ck_0^4 & \text{for a 2-d domain } \Omega, \\ C2^{2k_0} & \text{for a 3-d domain } \Omega, \end{cases}$$

where C depends on ϵ (which is fixed) and on other fixed parameters, but is independent of k_0. Then for $s = 1, 2, \ldots, [\frac{J}{k_0}]$ the following uniform estimates hold:

$$\|A^{(sk_0)}M^{(sk_0)}[\mathbf{v}]\|_{A^{(sk_0)^{-1}}} \leq \delta_2 \|\mathbf{v}\|_{A^{(sk_0)^{-1}}}$$

for a constant $\delta_2 \le C_1^{-1}(1+\epsilon)$, where C_1 is the constant from Theorem 3 (*related to the strengthened Cauchy inequality* (11)). *The latter represents the boundedness estimate. Similarly,*

$$\mathbf{v}^T M^{(sk_0)}[\mathbf{v}] \ge \delta_1 \mathbf{v}^T {A^{(sk_0)}}^{-1}\mathbf{v},$$

where $\delta_1 = \frac{1-\epsilon}{H_{k_0}}$, which represents the uniform coercivity estimate.

4.6. Complexity of the AMLI methods. To complete this large section, we must investigate the complexity of all stabilized HB multilevel preconditioners, i.e., the AMLI-type preconditioners from Definitions 7–10. Assume that we are in the setting (and the notation) of Theorem 3. Let n_k denote the number of degrees of freedom at the kth discretization level. We also assume uniform refinement. Then one has

$$\frac{n_{k+1}}{n_k} = 2^d + O(2^{-k}).$$

This implies that

$$n_{k+1} = O\left((2^d)^k n_1\right).$$

Let the cost of evaluating the action of ${B_{11}^{(k)}}^{-1}$ be of order $O(n_k - n_{k-1})$ arithmetic operations. Similarly, the actions of $A_{21}^{(k)}$ and $A_{12}^{(k)}$ require order $O(n_k - n_{k-1})$ operations and one action of $A^{(k)}$ has a cost of order $O(n_k)$ operations. Then, to implement one action of ${\tilde{B}^{(k)}}{}^{-1}$ (based on a polynomial $p_\nu(t)$ of degree ν), one must solve ν systems with $B^{(k)}$ and perform $\nu - 1$ actions of $A^{(k)}$. Denoting by $\mathcal{W}_s$ the cost of solving one system with $B^{(sk_0)}$, one then has the recurrence

$$\begin{aligned}
\mathcal{W}_{s+1} &\le \nu \mathcal{W}_s + C(n_{(s+1)k_0} - n_{sk_0}) + (\nu - 1) C n_{sk_0} \\
&\le \nu \mathcal{W}_s + C n_{sk_0} \\
&\le C \sum_{\sigma=0}^{s-1} \nu^\sigma n_{(s-\sigma+1)k_0} + \nu^s \mathcal{W}_1 \\
&= C \sum_{\sigma=0}^{s-1} \nu^\sigma \left(2^d\right)^{(s-\sigma+1)k_0 - 1} n_1 + \nu^s \mathcal{W}_1 \\
&= C n_1 \left(2^d\right)^{(s+1)k_0 - 1} \sum_{\sigma=0}^{s-1} \left(\frac{\nu}{2^{dk_0}}\right)^\sigma + \nu^s \mathcal{W}_1 \\
&\le n_{(s+1)k_0} \left[C \sum_{\sigma=0}^{s-1} \left(\frac{\nu}{2^{dk_0}}\right)^\sigma + \frac{\mathcal{W}_1}{n_{k_0}} \left(\frac{\nu}{2^{dk_0}}\right)^s \right].
\end{aligned}$$

Then, if $\frac{\nu}{2^{dk_0}} < 1$, one gets

$$\frac{\mathcal{W}_{s+1}}{n_{(s+1)k_0}} \le C + \frac{\mathcal{W}_1}{n_{k_0}}.$$

That is, the asymptotic work estimate shows that the AMLI preconditioners would be of optimal order if ν satisfied the inequality

$$\nu > C\sqrt{\eta_{k_0}} \quad \text{(from the spectral equivalence estimates; cf. Theorems 5 and 6)},$$

or for the variable-step AMLI preconditioner (cf. Theorem 7),

$$\nu > C H_{k_0}^2 = \begin{cases} C k_{k_0}^4 & \text{for a 2-d domain } \Omega, \\ C 2^{2k_0} & \text{for a 3-d domain } \Omega, \end{cases}$$

and for all AMLI preconditioners,

$$\frac{\nu}{2^{dk_0}} < 1 \quad \text{(from the complexity requirement).}$$

Based on the asymptotic behavior of η_{k_0} (see (10)), the restrictions on ν read as follows (except for the variable-step preconditioner):

$$2^{dk_0} > \nu > C\sqrt{\eta_{k_0}} = \begin{cases} O(\sqrt{k_0}), & d = 2, \text{ for } \Omega \text{ a plane polygon,} \\ O(2^{\frac{k_0}{2}}), & d = 3, \text{ for } \Omega \text{ a 3-d polytope.} \end{cases} \tag{20}$$

It is clear then that asymptotically, for k_0 sufficiently large, both inequalities for ν can be satisfied for both 2-d and 3-d problem domains.

For the variable-step AMLI preconditioner the relation between ν and k_0 reads as follows:

$$2^{dk_0} > \nu > \begin{cases} C k_0^4, & d = 2, \text{ for } \Omega \text{ a plane polygon,} \\ C 2^{2k_0}, & d = 3, \text{ for } \Omega \text{ a 3-d polytope.} \end{cases} \tag{21}$$

It is then again clear that for k_0 sufficiently large there is a ν such that relation (21) can be satisfied for both 2-d and 3-d problem domains.

Hence one may summarize as follows.

THEOREM 8. *The AMLI stabilized HB multilevel preconditioners from Definitions* 7–10 *give optimal order methods; that is, the corresponding preconditioned conjugate-gradient methods* (*variable-step conjugate-gradient methods in the case of Definition* 10) *have convergence rate bounded independently of the mesh size (or number of discretization levels) and one iteration step costs a number of arithmetic operations of order of the number of unknowns if, in general, k_0 is sufficiently large and ν (the polynomial degree or the number of inner conjugate-gradient iterations) is properly chosen with respect to k_0, that is, to satisfy relation* (20) *or* (21).

Since the AMLI preconditioners are implicitly defined and use recursive calls to a number of coarse levels, their implementation is a bit more involved. Implementation details can be found in Vassilevski [38], Axelsson and Vassilevski [8], and in Axelsson and Neytcheva [3], [4], and on massively parallel computers such as CM–200 in Neytcheva [30], [31].

5. Stabilizing the HB method, II: Approximate wavelets. There is an alternative way to stabilize the HB multilevel preconditioners. We have the option of changing the nodal interpolation operator Π_k. Similarly to the additive MG method (also called the BPX method; cf. Bramble, Pasciak, and Xu [14]), a good choice turns out to be the L^2-projection operators Q_k acting from $L^2(\Omega)$ to V_k defined by

$$(Q_k v, \psi) = (v, \psi) \quad \text{for all } \psi \in V_k.$$

Note that this involves the solution of mass matrix problems which are well conditioned. In what follows we will need only some good approximations to Q_k provided by a few steps of the polynomial iteration method applied to the above system. For example, if v has a local support, the approximation provided as just explained will also have a local support (depending upon the number of iterations performed with the given polynomial iteration method). See Figures 1 and 2.

The results here are based on a joint report of Vassilevski and Wang [41].

Now introduce the decomposition

$$V_k = (I - Q_{k-1})V_k + V_{k-1}.$$

Note that this is a direct decomposition. Also, observe that

$$V_k^1 \equiv (I - Q_{k-1})V_k = (I - Q_{k-1})(\Pi_k - \Pi_{k-1})V_k$$

since $(I - Q_{k-1})\Pi_{k-1} = 0$. That is,

$$V_k^1 = (I - Q_{k-1})V_k^{(1)},$$

which can be viewed as a modification of the HB component $V_k^{(1)} = (\Pi_k - \Pi_{k-1})V_k$ of V_k. The modification comes from the term $Q_{k-1}V_k^{(1)}$. That is, the difference with the HB decomposition is that we project in L^2-sense the HB component $V_k^{(1)}$ onto the next coarse space V_{k-1}. This provides us with a more stable decomposition of V. Specifically, we consider the decomposition

$$V = V_0 + V_1^1 + \cdots + V_J^1,$$

where $J \geq 1$ is the finest discretization level.

It is now more convenient to use operator function notation. To this end we define the following solution operators:

- $A^{(k)} \ : \ V_k \to V_k$ by

$$(A^{(k)}\psi, \theta) = A(\psi, \theta) \quad \text{for all } \psi, \theta \in V_k;$$

- $A_{11}^{(k)} \ : \ V_k^1 \to V_k^1$ by

$$(A_{11}^{(k)}\psi^1, \phi^1) = A(\phi^1, \psi^1) \quad \text{for all } \phi^1, \psi^1 \in V_k^1.$$

Similarly, we define the following operators:

- $A_{12}^{(k)} \ : \ V_{k-1} \to V_k^1$ and $A_{21}^{(k)} \ : \ V_k^1 \to V_{k-1}$ by

$$(A_{12}^{(k)}\tilde{\psi}, \phi^1) = A(\tilde{\psi}, \phi^1) \quad \text{for all } \tilde{\psi} \in V_{k-1} \text{ and all } \phi^1 \in V_k^1,$$
$$(A_{21}^{(k)}\phi^1, \tilde{\psi}) = A(\phi^1, \tilde{\psi}) \quad \text{for all } \phi^1 \in V_k^1 \text{ and all } \tilde{\psi} \in V_{k-1}.$$

Then the solution operator $A^{(k)}$ admits the two-by-two block form

$$A^{(k)} = \begin{array}{c} \left[\begin{array}{cc} A_{11}^{(k)} & A_{12}^{(k)} \\ A_{21}^{(k)} & A^{(k-1)} \end{array} \right] \end{array} \begin{array}{l} \} \quad V_k^1 \\ \} \quad V_{k-1} \end{array} .$$

5.1. Assumptions and preliminaries. We emphasize the following well-known stability estimate:

$$A(Q_kv, Q_kv) \leq \eta A(v, v) \quad \text{for any } v \in V_J \subset H_0^1(\Omega). \tag{22}$$

The constant η is uniformly bounded with respect to $(J - k) \to \infty$.

From now on we assume that the following basic norm equivalence estimate holds; namely,

- there exists a constant σ_N such that

$$|Q_0 v|_1^2 + \sum_{j=1}^{J} h_j^{-2} \|(Q_j - Q_{j-1})v\|_0^2 \le \sigma_N |v|_1^2 \quad \text{for any } v \in V = V_J. \tag{a.i}$$

The above estimate is shown, e.g., in Oswald [33]; see also Dahmen and Kunoth [16]. For a more detailed derivation of such stability estimates we refer the reader to Bornemann and Yserentant [11].

For the following analysis we will need the strengthened Cauchy–Schwarz inequality valid for entries in the finite element spaces V_i and V_j (see, e.g., Yserentant [44], Xu [42]; an equivalent result is also used in Vassilevski and Wang [40]):

- there exists a positive constant σ_I such that with $\delta = \frac{1}{\sqrt{2}}$ for any $i \le j$ there holds

$$(a(\psi_i, \psi_j))^2 \le \sigma_I \delta^{2(j-i)} a(\psi_i, \psi_i) \lambda_j \|\psi_j\|_0^2 \quad \text{for all } \psi_i \in V_i \text{ and } \psi_j \in V_j. \tag{a.ii}$$

Note that we have assumed $h_i = \frac{1}{2} h_{i-1}$ and $\lambda_j \equiv \lambda_{\max}[A^{(j)}] = O(h_j^{-2})$.

The following estimate plays a major role in the analysis of the method.

LEMMA 1. *Assume that* (a.i) *and* (a.ii) *hold. For any $v \in V_k$ and $s = k$ down to* 1, *denote $v^{(s)} = Q_s v$, $v_1^{(s)} = (Q_s - Q_{s-1})v$, and $v_2^{(s)} = v^{(s)} - v_1^{(s)} = Q_{s-1} v$. Then the following inequalities hold:*

$$\begin{aligned} \sum_{s=1}^{k} \lambda_s^{-1} \|A_{12}^{(s)} v_2^{(s)}\|_0^2 &\le \sum_{s=1}^{k} \lambda_s^{-1} \|A^{(s)} v_2^{(s)}\|_0^2 \\ &= \sum_{s=1}^{k} \lambda_s^{-1} \|A^{(s)} Q_{s-1} v\|_0^2 \\ &\le C A(v, v) \quad \textit{for any } v \in V_k. \end{aligned} \tag{23}$$

Proof. First note that $A_{12}^{(s)} = (Q_s - Q_{s-1}) A^{(s)} Q_{s-1}$. Hence, $\|A_{12}^{(s)} \tilde{\psi}\|_0 \le \|A^{(s)} \tilde{\psi}\|_0$ for any $\tilde{\psi} \in V_{s-1}$. Next, use the decomposition

$$v_2^{(j)} = Q_{j-1} v = \sum_{s=0}^{j-1} v_1^{(s)}, \quad v_1^{(s)} = (Q_s - Q_{s-1}) v \ (Q_{-1} = 0, \text{ i.e., } v_1^{(0)} = Q_0 v).$$

Introducing the operators

$$T_j = \lambda_j^{-1} A^{(j)},$$

we get the representation

$$\lambda_j^{-1} (A^{(j)} v_2^{(j)}, A^{(j)} v_2^{(j)}) = A(T_j v_2^{(j)}, v_2^{(j)}) = \sum_{s=0}^{j-1} A(T_j v_2^{(j)}, v_1^{(s)}).$$

Now, using the strengthened Cauchy–Schwarz inequality (a.ii) (note that $s \le j$), we get

$$\begin{aligned} |A(T_j v_2^{(j)}, v_1^{(s)})|^2 &\le \sigma_I^2 \delta^{2(j-s)} \ A(v_1^{(s)}, v_1^{(s)}) \ \lambda_j \|T_j v_2^{(j)}\|_0^2 \\ &= \sigma_I^2 \delta^{2(j-s)} \ A(v_1^{(s)}, v_1^{(s)}) \ \lambda_j^{-1} (A^{(j)} v_2^{(j)}, A^{(j)} v_2^{(j)}) \\ &= \sigma_I^2 \delta^{2(j-s)} \ A(v_1^{(s)}, v_1^{(s)}) \ A(T_j v_2^{(j)}, v_2^{(j)}). \end{aligned}$$

Therefore, substituting the last inequality into the preceding identity, we get

$$A(T_j v_2^{(j)}, v_2^{(j)}) \leq \sigma_I^2 \left[\sum_{s=0}^{j-1} \delta^{j-s} \left[A(v_1^{(s)}, v_1^{(s)}) \right]^{\frac{1}{2}} \right]^2 .$$

Applying the Cauchy–Schwarz inequality, we arrive at

$$A(T_j v_2^{(j)}, v_2^{(j)}) \leq \sigma_I^2 \frac{\delta}{1-\delta} \sum_{s=0}^{j-1} \delta^{j-s} A(v_1^{(s)}, v_1^{(s)}).$$

Summing over j yields

$$\begin{aligned}
\sum_{j=1}^{k} \lambda_j^{-1} \|A^{(j)} v_2^{(j)}\|_0^2 &= \sum_{j=1}^{k} A(T_j v_2^{(j)}, v_2^{(j)}) \\
&\leq \sigma_I^2 \frac{\delta}{1-\delta} \sum_{j=1}^{k} \sum_{s=0}^{j-1} \delta^{j-s} A(v_1^{(s)}, v_1^{(s)}) \\
&\leq \sigma_I^2 \left(\frac{\delta}{1-\delta} \right)^2 \sum_{s=0}^{k-1} A(v_1^{(s)}, v_1^{(s)}) \qquad (24) \\
&= \sigma_I^2 \left(\frac{\delta}{1-\delta} \right)^2 \Bigg[A(Q_0 v, Q_0 v) \\
&\qquad + \sum_{s=1}^{k-1} A((Q_s - Q_{s-1})v, (Q_s - Q_{s-1})v) \Bigg],
\end{aligned}$$

which together with the basic norm equivalence estimate (a.i) completes the proof.

Thus estimate (23) has been verified. ∎

5.2. Definition of the wavelet modified HB preconditioner.

Definition 11 (multiplicative or block Gauss–Seidel wavelet modified HB multilevel preconditioner). *Let $M^{(0)} = A^{(0)}$. For $k \geq 1$,*

$$M^{(k)} = \begin{bmatrix} B_{11}^{(k)} & 0 \\ A_{21}^{(k)} & M^{(k-1)} \end{bmatrix} \begin{bmatrix} I & {B_{11}^{(k)}}^{-1} A_{12}^{(k)} \\ 0 & I \end{bmatrix} \begin{array}{l} \} \;\; V_k^1 \\ \} \;\; V_{k-1} \end{array} .$$

Here $B_{11}^{(k)}$ are given symmetric positive-definite approximations to the solution operators $A_{11}^{(k)}$ defined on the spaces V_k^1.

Remark 1. The difficulty with the above preconditioner from Definition 11 is that there is no computationally feasible basis of V_k^1 since the wavelet bases for finite element spaces have nonlocal support. Hence a natural step is instead to use approximate L^2-projection operators Q_k^a. Then $(I - Q_{k-1}^a)(\Pi_k - \Pi_{k-1})\phi^1$, when ϕ^1 runs over the nodal basis of $V_k^{(1)} = (\Pi_k - \Pi_{k-1})V_k$, will form a basis of V_k^1 with locally supported functions if $Q_{k-1}^a \phi^1$ has a local support. This will be the case if $Q_{k-1}^a \phi^1$ corresponds to a fixed number of iterations of the polynomial iterative method for solving the mass matrix equation

$$(Q_{k-1}\phi^1, \theta) = (\phi^1, \theta) \quad \text{for all } \theta \in V_{k-1}.$$

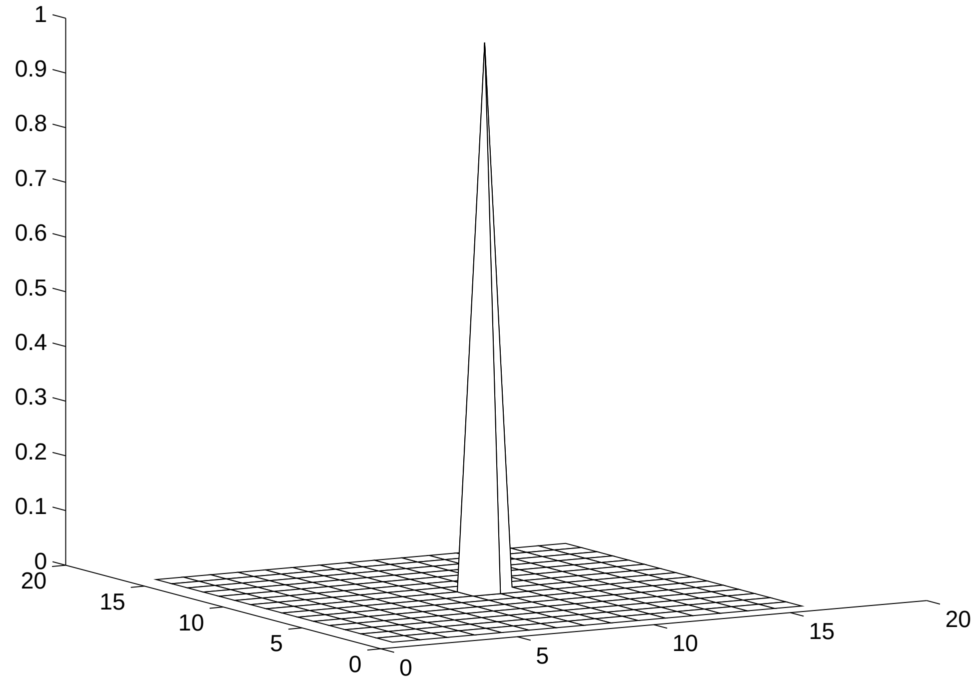

FIG. 1. *Plot of an HB function (no modification).*

In Figure 1 we show a typical plot of a nodal basis function ϕ^1. Its wavelet-like modification, obtained by approximately solving the above mass matrix equation using $m = 2$ steps of the conjugate-gradient method, is shown in Figure 2.

5.3. Approximate wavelet modified HB methods. Here we assume that there is an approximation Q_k^a of Q_k such that

$$\|(Q_k - Q_k^a)v\|_{L^2(\Omega)} \leq \tau \|Q_k v\|_{L^2(\Omega)} \quad \text{for all } v \in L^2(\Omega). \tag{25}$$

The constant τ is assumed sufficiently small (see (27) below). We stress that τ is assumed independent of the mesh size or of the number of levels J.

We consider the spaces

$$\begin{aligned} V_k^1 &= (I - Q_{k-1}^a)(\Pi_k - \Pi_{k-1})V_k \\ &= (I - Q_{k-1}^a)(\Pi_k - \Pi_{k-1})V. \end{aligned}$$

We have the two-level decomposition

$$V_k = V_k^1 + V_{k-1}.$$

That is, $v = (I - Q_{k-1}^a)(\Pi_k - \Pi_{k-1})v + \left[Q_{k-1}^a + (I - Q_{k-1}^a)\Pi_{k-1}\right] v$ for any $v \in V_k$. On the basis of the pair of spaces V_k^1 and V_{k-1} we define the preconditioner $M^{(k)}$ as it was defined in Definition 11. To analyze the method we need some auxiliary estimates. Define $v_1^{(s)} = (I - Q_{s-1}^a)(\Pi_s - \Pi_{s-1})v^{(s)}$ and $v^{(s-1)} = v_2^{(s)} = \left[Q_{s-1}^a + (I - Q_{s-1}^a)\Pi_{s-1}\right] v^{(s)}$ starting with $v^{(J)} = v$ for any given $v \in V$ and $s = J$ down to 1.

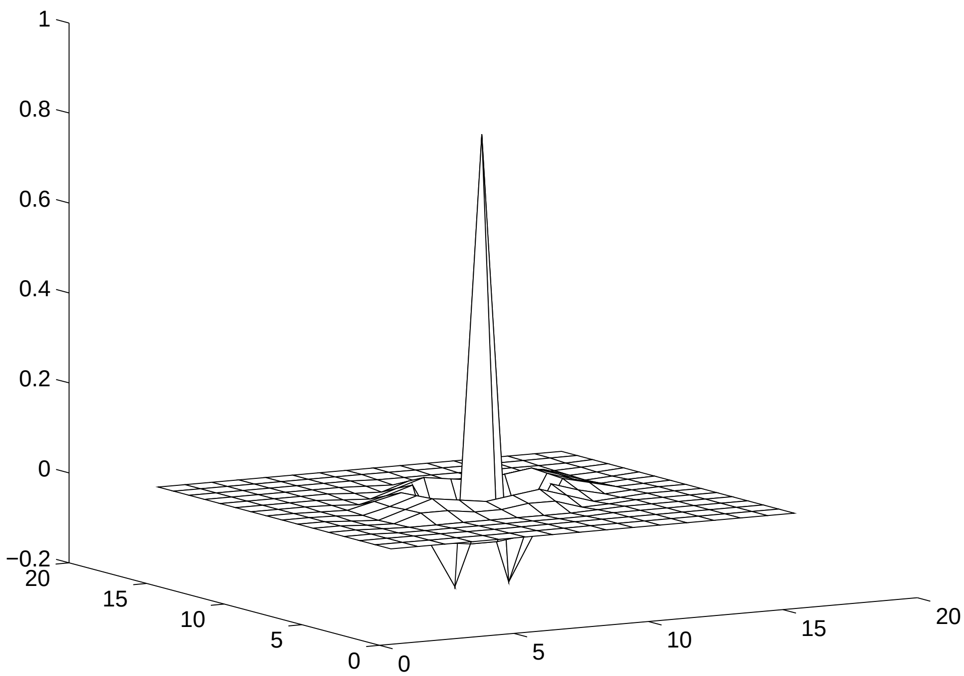

FIG. 2. *Plot of a wavelet modified HB function; $m = 2$.*

We have for any $v \in V_k$,

$$((M^{(k)} - A^{(k)})v, v) = ((B_{11}^{(k)} - A_{11}^{(k)})v_1^{(k)}, v_1^{(k)}) + ((M^{(k-1)} - A^{(k-1)})v^{(k-1)}, v^{(k-1)})$$
$$+(B_{11}^{(k)^{-1}} A_{12}^{(k)} v_2^{(k)}, A_{12}^{(k)} v_2^{(k)}).$$

This identity at first implies by induction (since $M^{(0)} = A^{(0)}$) that $M^{(k)} - A^{(k)}$ is positive semidefinite. Using it recursively, one arrives at the major inequality (cf. (8))

$$\begin{aligned} ((M^{(k)} - A^{(k)})v, v) &\leq b_1(A_{11}^{(k)} v_1^{(k)}, v_1^{(k)}) + ((M^{(k-1)} - A^{(k-1)})v^{(k-1)}, v^{(k-1)}) \\ &\quad +(A_{11}^{(k)^{-1}} A_{12}^{(k)} v_2^{(k)}, A_{12}^{(k)} v_2^{(k)}) \\ &\leq b_1 \sum_{s=1}^{k} (A_{11}^{(s)} v_1^{(s)}, v_1^{(s)}) + \sum_{s=1}^{k} (A_{11}^{(s)^{-1}} A_{12}^{(s)} v_2^{(s)}, A_{12}^{(s)} v_2^{(s)}). \end{aligned} \tag{26}$$

5.4. Estimation of the deviation from the exact wavelet decomposition. We next estimate the deviation $e_s = v^{(s)} - Q_s v$. The following recursive relation holds (cf. Vassilevski and Wang [41]):

$$e_{s-1} = [Q_{s-1} + R_{s-1}]e_s + R_{s-1}(Q_s - Q_{s-1})v, \quad \text{where}$$
$$R_{s-1} = (Q_{s-1} - Q_{s-1}^a)(\Pi_{s-1} - \Pi_s).$$

It is not as hard to estimate the L^2-norm of e_s. The L^2-norm is denoted in what follows by $\|.\|_0$. We have, for any $\phi \in V_s$,

$$\|R_{s-1}\phi\|_0 \leq \tau\|(\Pi_s - \Pi_{s-1})\phi\|_0 \leq C_R\tau\|\phi\|_0.$$

Here C_R is a positive constant independent of s and J. We only remark at this point that C_R in practice can be estimated by an element-by-element analysis (with respect to the elements of $\mathcal{T}_{s-1}$).

Therefore,

$$\|e_{s-1}\|_0 \le (1 + C_R\tau)\|e_s\|_0 + C_R\tau\|(Q_s - Q_{s-1})v\|_0.$$

LEMMA 2. *Assume that* (a.i) *and* (a.ii) *hold. Let* $\lambda_k = O(h_k^{-2}) = O(2^{2k})$ *be an estimate of the maximum eigenvalue of the operator* $A^{(k)}$. *Assume also that* τ *is sufficiently small such that*

$$C_R\tau \le q_1 = \text{ const } < 1; \tag{27}$$

that is,

$$(1 + C_R\tau)\frac{1}{2} \le q = \frac{1+q_1}{2} = \text{ const } < 1. \tag{27$'$}$$

Then the following major estimate holds:

$$\sum_{s=1}^{k} \lambda_{s-1}\|e_{s-1}\|_0^2 \le C\tau^2 A(v,v) \quad \textit{for any } v \in V_k. \tag{28}$$

Proof. Using the fact that $e_k = 0$ (since $v \in V_k$), by simple recurrence we obtain

$$\|e_{s-1}\|_0 \le C_R\tau \sum_{j=s}^{k} (1 + C_R\tau)^{j-s}\|(Q_j - Q_{j-1})v\|_0.$$

Therefore, assuming that $h_j = \frac{1}{2}h_{j-1}$ and using (27$'$) and the Cauchy–Schwarz inequality, we get

$$\begin{aligned}
\|e_{s-1}\|_0 &\le C_R\tau\ h_{s-1} \sum_{j=s}^{k} (1 + C_R\tau)^{j-s} h_s^{-1}\|(Q_j - Q_{j-1})v\|_0 \\
&= C_R\tau\ h_{s-1} \sum_{j=s}^{k} (1 + C_R\tau)^{j-s} h_s^{-1} h_j h_j^{-1}\|(Q_j - Q_{j-1})v\|_0 \\
&= C_R\tau\ h_{s-1} \sum_{j=s}^{k} (1 + C_R\tau)^{j-s} \left(\frac{1}{2}\right)^{j-s} h_j^{-1}\|(Q_j - Q_{j-1})v\|_0 \\
&\le C_R\tau\ h_{s-1} \sum_{j=s}^{k} q^{j-s} h_j^{-1}\|(Q_j - Q_{j-1})v\|_0 \\
&\le C_R\tau\ h_{s-1} \frac{1}{\sqrt{1-q}} \left[\sum_{j=s}^{k} q^{j-s} h_j^{-2}\|(Q_j - Q_{j-1})v\|_0^2\right]^{\frac{1}{2}}.
\end{aligned}$$

The latter inequality shows that

$$\begin{aligned}
\sum_{s=1}^{k} h_{s-1}^{-2}\|e_{s-1}\|_0^2 &\le C_R^2\tau^2 \frac{1}{1-q} \sum_{s=1}^{k}\sum_{j=s}^{k} q^{j-s} h_j^{-2}\|(Q_j - Q_{j-1})v\|_0^2 \\
&\le C_R^2\tau^2 \frac{1}{(1-q)^2} \sum_{j=1}^{k} h_j^{-2}\|(Q_j - Q_{j-1})v\|_0^2 \\
&\le C\tau^2 \frac{1}{(1-q)^2} A(v,v).
\end{aligned}$$

Here we used the estimate (a.i). Note that $\frac{1}{1-q} = \frac{2}{1-C_R\tau}$ if we let $q = \frac{1+C_R\tau}{2}$. ∎

5.5. Analysis of the approximate wavelet modified HB preconditioner. To complete the analysis of the method we must estimate the two sums in the last line of (26).

To this end, consider $v_1^{(s)} = v^{(s)} - v^{(s-1)} = (Q_s - Q_{s-1})v + e_s - e_{s-1}$. Using (28), we immediately find that the first sum in the last inequality of (26) can be estimated as follows:

$$\begin{aligned}\sum_{s=1}^{k}(A_{11}^{(s)}v_1^{(s)}, v_1^{(s)}) &\leq \sum_{s=1}^{k}\lambda_s \|v_1^{(s)}\|_0^2 \\ &\leq 3\sum_{s=1}^{k}\lambda_s\left(\|(Q_s - Q_{s-1})v\|_0^2 + \|e_s\|_0^2 + \|e_{s-1}\|_0^2\right) \\ &\leq C(\|v\|_1^2 + A(v,v)) \\ &\leq CA(v,v).\end{aligned}$$

Here we have again used the norm equivalence estimate $C\|\phi\|_1^2 \geq \sum_{s=1}^{k}\lambda_s\|(Q_s - Q_{s-1})\phi\|_0^2$ (i.e., estimate (a.i)). The final estimate that we need in (26) reads as follows:

$$\begin{aligned}\sum_{s=1}^{k}({A_{11}^{(s)}}^{-1}A_{12}^{(s)}v^{(s-1)}, A_{12}^{(s)}v^{(s-1)}) &\leq C\sum_{s=1}^{k}h_s^2\|A_{12}^{(s)}v^{(s-1)}\|_0^2 \\ &\leq C\sum_{s=1}^{k}h_s^2\|A^{(s)}v^{(s-1)}\|_0^2 \\ &\leq C\sum_{s=1}^{k}h_s^2\left(\|A^{(s)}e_{s-1}\|_0^2 + \|A^{(s)}Q_{s-1}v\|_0^2\right) \\ &\leq C\sum_{s=1}^{k}\lambda_{s-1}\|e_{s-1}\|_0^2 + C\sum_{s=1}^{k}\lambda_s^{-1}\|A^{(s)}Q_{s-1}v\|_0^2 \\ &\leq CA(v,v).\end{aligned}$$

Here we have used Lemma 2 (estimate (28)), the fact that the first blocks $A_{11}^{(s)}$ are well conditioned (that $\lambda_{\min}[A_{11}^{(s)}] = O(h_s^{-2})$—note that $\lambda_{\max}[A_{11}^{(s)}] = O(h_s^{-2})$), and the major estimate from Lemma 1,

$$\sum_{s=1}^{k}\lambda_s^{-1}\|A^{(s)}Q_{s-1}v\|_0^2 \leq CA(v,v). \tag{29}$$

The fact that $A_{11}^{(s)}$ are well conditioned was proven in Vassilevski and Wang [41]. Estimate (29) (also proven in [41]) assumes (a.i) and (a.ii). It is straightforward, however, to prove a suboptimal estimate without those assumptions. One has

$$\sum_{s=1}^{k}\lambda_s^{-1}\|A^{(s)}Q_{s-1}v\|_0^2 \leq \sum_{s=1}^{k}A(Q_{s-1}v, Q_{s-1}v) \leq k\eta A(v,v).$$

Here η stands for the uniform $A(.,.)$-norm bound of any of the L^2-projection operators Q_s (see (22)). Therefore, we can formulate the following main result.

THEOREM 9. *The approximate wavelet modified HB multiplicative preconditioner $M^{(k)}$ as defined in Definition 11 gives a spectrally equivalent preconditioner to $A^{(k)}$ if the approximate L^2-projections are accurate enough (e.g., such that (27) holds, the bound of which is independent of the mesh size or J). This holds provided assumptions (a.i) and (a.ii) hold. Without assuming (a.i) and (a.ii), $M^{(k)}$ is proven to be only nearly spectrally equivalent to $A^{(k)}$. The preconditioner can be implemented such that one action of $M^{-1} = {M^{(J)}}^{-1}$ requires $O(n \log \tau^{-1}) = O(n)$ arithmetic operations (since τ is independent of J or the mesh size). Here, $n = n_J$ is the number of the total degrees of freedom.*

5.6. Additive version of the approximate wavelet modified HB preconditioner. Finally, one can consider the additive version of the approximate wavelet modified HB preconditioner, which is defined as follows.

DEFINITION 12 (additive approximate wavelet modified HB preconditioner). *Set $M_D^{(0)} = A^{(0)}$ and for $k = 1, 2, \ldots, J$ define*

$$M_D^{(k)} = \begin{array}{c} \left[\begin{array}{ccccc} D_{11}^{(k)} & 0 & & & 0 \\ 0 & D_{11}^{(k-1)} & 0 & & \\ & \ddots & \ddots & \ddots & \\ & & 0 & D_{11}^{(1)} & 0 \\ 0 & & & 0 & A^{(0)} \end{array} \right] \end{array} \begin{array}{l} \} \;\; V_k^1 \\ \} \;\; V_{k-1}^1 \\ \vdots \\ \} \;\; V_1^1 \\ \} \;\; V_0 \end{array} .$$

Here $D_{11}^{(k)}$ is, for example, the diagonal part of $A_{11}^{(k)}$.

It was shown in Vassilevski and Wang [41] that if τ in (25) is sufficiently small, independent of the mesh size or J, the additive version of the approximate wavelet modified HB multilevel preconditioner $M_D^{(k)}$ is spectrally equivalent to the corresponding solution operator $A^{(k)}$. Here, assumptions (a.i) and (a.ii) are again needed. We conclude with the following result.

THEOREM 10. *The additive version of the approximate wavelet modified HB multilevel preconditioner $M_D^{(k)}$ as defined in Definition 12 is spectrally equivalent to $A^{(k)}$ if (25) holds with a sufficiently small constant τ (independent of the mesh size or J). The method can be implemented such that one action of $M_D^{-1} = {M_D^{(J)}}^{-1}$ requires $O(n \log \tau^{-1}) = O(n)$ arithmetic operations ($n = n_J$ is the number of the total degrees of freedom); i.e., the method is optimal.*

5.7. Concluding remarks. Implementation details together with some numerical results for both the multiplicative and additive approximate wavelet modified HB multilevel preconditioners can be found in Vassilevski and Wang [41].

We remark that related results for constructing multilevel methods based on direct decompositions of finite element spaces can be found in Stevenson [35] and Griebel and Oswald [20]. These methods deal with tensor product meshes and exploit 1-d wavelet space decompositions, and therefore cannot handle more general triangulations. In Stevenson's [36] some progress was made toward more general meshes. The method from Vassilevski and Wang [41] handles the general case; it applies whenever HB decomposition of the finite element space exists.

Acknowledgments. Thanks are due to Dr. Svetozar Margenov, who critically read a preliminary version of the paper, and also to the referees for their helpful suggestions.

REFERENCES

[1] B. Achchab and J.-F. Maitre, *Estimate of the constant in two strengthened C.B.S. inequalities for FEM systems of 2D elasticity: Application to multilevel methods and a posteriori error estimators*, Numer. Linear Algebra Appl., 3 (1996), pp. 147–159.

[2] O. Axelsson and I. Gustafsson, *Preconditioning and two-level multigrid methods of arbitrary degree of approximations*, Math. Comp., 40 (1983), pp. 219–242.

[3] O. Axelsson and M. G. Neytcheva, *Algebraic multilevel iteration methods for Stieltjes matrices*, Numer. Linear Algebra Appl., 1 (1994), pp. 213–236.

[4] O. Axelsson and M. G. Neytcheva, *The Short Length AMLI*, I, Report # 9417-94, Department of Mathematics, University of Nijmegen, The Netherlands, 1994.

[5] O. Axelsson and P. S. Vassilevski, *Algebraic multilevel preconditioning methods*, I, Numer. Math., 56 (1989), pp. 157–177.

[6] O. Axelsson and P. S. Vassilevski, *Algebraic multilevel preconditioning methods*, II, SIAM J. Numer. Anal., 27 (1990), pp. 1569–1590.

[7] O. Axelsson and P. S. Vassilevski, *A black box generalized conjugate gradient solver with inner iterations and variable-step preconditioning*, SIAM J. Matrix Anal. Appl., 12 (1991), pp. 625–644.

[8] O. Axelsson and P. S. Vassilevski, *Variable-step multilevel preconditioning methods*, Numer. Linear Algebra Appl., 1 (1994), pp. 75–101.

[9] R. E. Bank and T. Dupont, *Analysis of a Two-level Scheme for Solving Finite Element Equations*, Report CNA-159, Center for Numerical Analysis, The University of Texas at Austin, Austin, TX, 1980.

[10] R. E. Bank, T. Dupont, and H. Yserentant, *The hierarchical basis multigrid method*, Numer. Math., 52 (1988), pp. 427–458.

[11] F. Bornemann and H. Yserentant, *A basic norm equivalence for the theory of multilevel methods*, Numer. Math., 64 (1993), pp. 455–476.

[12] D. Braess, *The contraction number of a multigrid method for solving the Poisson equation*, Numer. Math., 37 (1981), pp. 387–404.

[13] J. Bramble, *Multigrid Methods*, 2nd ed., Pitman Research Notes in Mathematics Series, # 234, Longman Scientific and Technical, Harlow, UK, 1995.

[14] J. H. Bramble, J. E. Pasciak, and J. Xu, *Parallel multilevel preconditioners*, Math. Comp., 55 (1990), pp. 1–22.

[15] T. F. Chan and T. Mathew, *Domain decomposition algorithms*, Acta Numerica, (1994), pp. 61–143.

[16] W. Dahmen and A. Kunoth, *Multilevel preconditioning*, Numer. Math., 63 (1992), pp. 315–344.

[17] M. Dryja, B. Smith, and O. B. Widlund, *Schwarz analysis of iterative substructuring algorithms for elliptic problems in three dimensions*, SIAM J. Numer. Anal., 31 (1994), pp. 1662–1694.

[18] V. L. Eijkhout and P. S. Vassilevski, *The role of the strengthened C.B.S. inequality in multilevel methods*, SIAM Rev., 33 (1991), pp. 405–419.

[19] M. Griebel and P. Oswald, *On the abstract theory of additive and multiplicative Schwarz algorithms*, Numer. Math., 70 (1995), pp. 163–180.

[20] M. Griebel and P. Oswald, *Tensor product type subspace splittings and multilevel iterative methods for anisotropic problems*, Adv. Comput. Math., 4 (1994), pp. 171–206.

[21] X.-Z. Guo, *The Algebraic Hierarchical Basis Multigrid Method*, Institute for Advanced Computer Studies Report, Univ. of Maryland, College Park, MD, 1992.

[22] W. Hackbusch, *Multigrid Methods and Applications*, Springer-Verlag, Berlin, New York, 1985.

[23] Y. A. Kuznetsov, *Algebraic multigrid domain decomposition methods*, Soviet J. Numer. Meth. and Math. Modeling, 4 (1989), pp. 351–380.

[24] J. F. Maitre and F. Musy, *The contraction number of a class of two-level methods; an exact evaluation for some finite element subspaces and model problems*, in Multigrid Methods, Proceedings, W. Hackbusch and U. Trottenberg, eds., Lecture Notes in Math. # 960, Springer-Verlag, Berlin, New York, 1982, pp. 201–216.

[25] S. D. Margenov, *Upper bound of the constant in the strengthened C.B.S. inequality for FEM 2D elasticity equations*, Numer. Linear Algebra Appl., 1 (1994), pp. 65–74.

[26] S. D. Margenov and P. S. Vassilevski, *Algebraic multilevel preconditioning of anisotropic elliptic problems*, SIAM J. Sci. Comput., 15 (1994), pp. 1026–1037.

[27] S. Margenov, L. Xanthis, and L. Zikatanov, *On the optimality of the semi-coarsening AMLI algorithm*, in Iterative Methods in Linear Algebra, II, IMACS Series in Computational and Applied Mathematics, Vol. 3 (Proc. of the 2nd IMACS International Symposium

on Iterative Linear Algebra, June 17–20, 1995, Blagoevgrad, Bulgaria), S. D. Margenov and P. S. Vassilevski, eds., IMACS, Rutgers University – Department of Computer Science, Piscataway, NJ, 1996, pp. 270–280.

[28] J. M. Maubach, *Local bisection refinement for n-simplicial grids by reflection*, SIAM J. Sci. Comput., 16 (1995), pp. 210–227.

[29] W. F. Mitchell, *Optimal multilevel iterative methods for adaptive grids*, SIAM J. Sci. Stat. Comput., 13 (1992), pp. 146–167.

[30] M. G. Neytcheva, *Experience in implementing the algebraic multilevel iteration method on a SIMD-type computer*, Appl. Numer. Math., 19 (1995), pp. 71–90.

[31] M. G. Neytcheva, *Arithmetic and communication complexity of preconditioning methods*, Ph.D. thesis, Faculty of Mathematics and Informatics, University of Nijmegen, Nijmegen, The Netherlands, 1995.

[32] M.-E. G. Ong, *Hierarchical basis preconditioners in three dimensions*, SIAM J. Sci. Comput., 18 (1997), to appear.

[33] P. Oswald, *Multilevel Finite Element Approximation. Theory and Applications*, Teubner Skripten zur Numerik, Teubner, Stuttgart, 1994.

[34] J. W. Ruge and K. Stüben, *Algebraic multigrid*, in Multigrid Methods, S. McCormick, ed., SIAM, Frontiers in Applied Mathematics, Philadelphia, PA, 1987, pp. 73–130.

[35] R. Stevenson, *Robustness of the additive and multiplicative frequency decomposition multilevel method*, Computing, 54 (1995), pp. 331–346.

[36] R. Stevenson, *A Robust Hierarchical Basis Preconditioner on General Meshes*, Report # 9533, Department of Mathematics, University of Nijmegen, Nijmegen, The Netherlands, 1995.

[37] P. S. Vassilevski, *Nearly Optimal Iterative Methods for Solving Finite Element Elliptic Equations Based on the Multilevel Splitting of the Matrix*, Report # 1989-09, Institute for Scientific Computation, University of Wyoming, Laramie, WY, 1989.

[38] P. S. Vassilevski, *Hybrid V-cycle algebraic multilevel preconditioners*, Math. Comp., 58 (1992), pp. 489–512.

[39] P. S. Vassilevski and M. H. Etova, *Computation of constants in the strengthened Cauchy inequality for elliptic bilinear forms with anisotropy*, SIAM J. Sci. Stat. Comput., 13 (1992), pp. 655–665.

[40] P. S. Vassilevski and J. Wang, *An application of the abstract multilevel theory to nonconforming finite element methods*, SIAM J. Numer. Anal., 32 (1995), pp. 235–248.

[41] P. S. Vassilevski and J. Wang, *Stabilizing the Hierarchical Basis by Approximate Wavelets,* I: *Theory, and* II: *Implementation and Numerical Results*, CAM Reports 95-47 and 95-48, Department of Mathematics, UCLA, Los Angeles, CA, 1995.

[42] J. Xu, *Iterative methods by space decomposition and subspace correction*, SIAM Rev., 34 (1992), pp. 581–613.

[43] H. Yserentant, *On the multilevel splitting of finite element spaces*, Numer. Math., 49 (1986), pp. 379–412.

[44] H. Yserentant, *Old and new convergence proofs for multigrid algorithms*, Acta Numerica, (1993), pp. 285–326.